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# Relative $L^p$ and Orlicz cohomology and Applications to Heintze groups

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by

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**Abstract**

This work has two parts. In the first we define the $L^p$-cohomology of certain Gromov-hyperbolic spaces relative to a point on its boundary at infinity. This is done in two different contexts. First we consider a simplicial version, defined for simplicial complexes with bounded geometry. In a similar way as in the classical case we prove the quasi-isometry invariance under a contractibility condition. Then we define a relative version of the de Rham $L^p$-cohomology in the case of Riemannian manifolds. We study the relationship between these two definitions, which allows to conclude that this second version is also invariant under certain hypothesis. As an application we study the $L^p$-cohomology relative to a special point on the boundary of Heintze groups of the form $\mathbb{R}^{n-1} \rtimes_\alpha \mathbb{R}$, where the derivation $\alpha$ has positive eigenvalues $\lambda_1 \leq \cdots \leq \lambda_{n-1}$. As a consequence the numbers $\frac{\lambda_1}{\mathrm{tr}(\alpha)}, \ldots, \frac{\lambda_{n-1}}{\mathrm{tr}(\alpha)}$ are invariant by quasi-isometries.

In the second part we work with Orlicz cohomology, which is a generalization of $L^p$-cohomology. We also define a relative version and adapt the proof of the quasi-isometry invariance in the simplicial case. As the main result of this part we prove the equivalence between the simplicial (relative) Orlicz cohomology and the (relative) Orlicz-de Rham cohomology for Lie groups. An important consequence of this is the quasi-isometry invariance of Orlicz-de Rham cohomology in the case of contractible Lie groups.

## Resumen

Este trabajo consta de dos partes. En la primera se define la cohomología $L^p$ de ciertos espacios métricos Gromov-hiperbólicos relativa a un punto de su borde al infinito. Esto se hace en dos diferentes contextos. Primero se desarrolla una versión simplicial, definida para complejos simpliciales de geometría acotada. Se prueba aquí, al igual que como se hace en el caso clásico, que esta es invariante por cuasi-isometrías bajo cierta condición de contractibilidad. Luego se define una versión relativa de la cohomología $L^p$ de De Rham en el caso de variedades Riemannianas. Se estudia la relación entre estas dos definiciones, lo que permite concluir que también esta segunda versión es invariante por cuasi-isometrías bajo ciertas hipótesis. Como aplicación de lo anterior se estudia la cohomología $L^p$ relativa a un punto distinguido en el borde de los grupos de Heintze de la forma $\mathbb{R}^{n-1} \rtimes_\alpha \mathbb{R}$, donde la derivación $\alpha$ tiene valores propios reales positivos $\lambda_1 \leq \cdots \leq \lambda_{n-1}$. Como consecuencia se obtiene que los números $\frac{\lambda_1}{\mathrm{tr}(\alpha)}, \ldots, \frac{\lambda_{n-1}}{\mathrm{tr}(\alpha)}$ son invariantes por cuasi-isometrías.

En la segunda parte se trabaja con la cohomología de Orlicz, que es una generalización de la cohomología $L^p$. Aquí también se define una versión relativa y se adapta la prueba de la invarinza por cuasi-isometrías de la cohomología de Orlicz simplicial. Como resultado central de esta segunda parte se prueba la equivalencia entre la cohomología de Orlicz simplicial (relativa) y la cohomología de Orlicz-de Rham (relativa) para grupos de Lie. Una importante consecuencia de esto es la invarianza por cuasi-isometrías de la cohomología de Orlicz-de Rham en el caso de los grupos de Lie contractibles.

**Résumé**

Ce texte est divisé en deux parties. Dans la première on définit la cohomologie $L^p$ de certains espaces métriques hyperboliques d'après Gromov relativement à un point dans son bord à l'infini. Deux aspects différents sont traités. En premier on étudie une version simpliciale de la cohomologie $L^p$ adaptée aux complexes simpliciaux à géométrie bornée. On montre, de manière similaire au cas classique, qu'elle est invariante par quasi-isométries sous certaines hypothèses. Ensuite on définit une version relative de la cohomologie $L^p$ de De Rham dans le cas des variétés riemanniennes. On étudie la relation entre ces deux notions, on en déduit que la deuxième version est aussi invariante par quasi-isometries sous certaines hypothèses. Comme application on étudie la cohomologie $L^p$ relative à un point distingué dans le bord des groupes d'Heintze $\mathbb{R}^{n-1} \rtimes_\alpha \mathbb{R}$, où la dérivation $\alpha$ a toutes ses valeurs propres réelles positives $\lambda_1 \leq \cdots \leq \lambda_{n-1}$. Comme conséquence on obtient que les nombres $\frac{\lambda_1}{\mathrm{tr}(\alpha)}, \ldots, \frac{\lambda_{n-1}}{\mathrm{tr}(\alpha)}$ sont invariants par quasi-isometries.

Dans la deuxième partie on travaille avec la cohomologie d'Orlicz, une généralisation de la cohomologie $L^p$. On définit aussi une version relative et on adapte la preuve de l'invariance par quasi-isometries de la cohomologie d'Orlicz simpliciale. Comme résultat central de cette deuxième partie on démontre l'équivalence entre la cohomologie d'Orlicz simpliciale (relative) et la cohomologie d'Orlicz-de Rham (relative) pour les groupes de Lie. Une conséquence importante est l'invariance par quasi-isometries de la cohomologie d'Orlicz-de Rham dans le cas des groupes de Lie contractiles.

# Contents

# Chapter 1

# Introduction

## 1.1 Motivation

Consider the following fundamental problem of large scale geometry:

**Problem 1.1.1.** *Given a family of metric spaces, how to determine its quasi-isometry classes?*

In this context the $L^p$-cohomology in its different versions appears as an important quasi-isometry invariant, and as a consequence, as a tool to give partial answers to Problem 1.1.1.

To give a quick idea remember the classical de Rham cohomology of a smooth manifold $M$. It is defined from the cochain complex of differential forms

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \xrightarrow{d} \cdots$$

and provides a topological invariant. That is, if $M$ and $N$ are diffeomorphic, there is an isomorphism between the cohomology groups $H^k_{\mathrm{dR}}(M)$ and $H^k_{\mathrm{dR}}(N)$ for every $k \in \mathbb{N}$.

Demanding an integrability condition to the forms on some Riemannian manifold we put the metric in the game. One can consider, for example, the spaces of differential forms that are $L^p$-integrable and have $L^p$-integrable derivative. This cochain complex defines the de Rham $L^p$-cohomology, introduced in the eighties ([GKS87, Pan88]). It is possible also to consider the space of $L^p$-integrable differential forms with $L^q$-integrable derivative for another positive number $q \geq 1$, which defines the de Rham $L^{p,q}$-cohomology (we refer to [GT06] for more details about this second notion, that will not be studied in this work).

Under certain hypotheses one can prove that the $L^p$-cochain complex described above is homotopically equivalent to another cochain complex: the one that consists of simplicial cochains on a proper triangulation of the manifold that have finite $\ell^p$-norm (see [GKS88, Pan95, Gen14]). The cohomology of this cochain complex is called the simplicial $\ell^p$-cohomology of the triangulation. We refer also to [Ele97, BP03] for this version.

Other versions of $L^p$-cohomology have been studied. For instance, we can mention the *Alexander-Spanier* and *asymptotic $L^p$-cohomology*, defined for metric measure spaces ([Pan95, Gen14]); or the *continuous group $L^p$-cohomology* ([CT11, BR19]).

As we said, $L^p$-cohomology provides a quasi-isometry invariant: If $M$ and $N$ are two quasi-isometric Riemannian manifolds with certain properties (uniformly contractible, bounded geometry), then for every $k \in \mathbb{N}$ the $L^p$-cohomology spaces $L^pH^k(M)$ and $L^pH^k(N)$ are isomorphic as topological vector spaces; or in its simplicial version, if $X$ and $Y$ are two simplicial complexes equipped with certain metrics, then the $\ell^p$-cohomology spaces $\ell^pH^k(X)$ and $\ell^pH^k(Y)$ are isomorphic (see Section 1.2.1 for the explicit formulations). This result appears first for degree one in [Pan88]. Later it is generalized to higher degrees ([Gro93, Pan95, BP03]).

Since different versions of $L^p$-cohomology are quasi-isometry invariant restricted to the respective family of metric spaces, their properties are also invariant. In particular some numerical invariants can be obtained by studying the $L^p$-cohomology spaces. For example, if we consider the de Rham $L^p$-cohomology of a Riemannian manifold $M$, the sets

$$\begin{aligned}&\mathrm{Ann}_k(M) = \{p \in [1, +\infty) : L^pH^k(M) = 0\} \text{ and}\\&\mathrm{Haus}_k(M) = \{p \in [1, +\infty) : L^pH^k(M) \text{ is a Hausdorff space}\}\end{aligned}$$

are also quasi-isometry invariant. In some cases the sets $\mathrm{Ann}_k(M)$ and $\mathrm{Haus}_k(M)$ are finite collections of intervals whose ends are numerical invariants.

In case of Gromov-hyperbolic spaces there exists an identification between the $L^p$-cohomology spaces in degree one and some Besov spaces defined on the boundary at infinity, that allows to study the sets $\mathrm{Haus}_1$ and $\mathrm{Ann}_1$ (see [Pan88, BP03]). In degree $k \geq 2$ the techniques are quite different. In [Pan08] and [Bou16] there are proofs of the vanishing of the $L^p$-cohomology via the explicit construction of primitives, which gives a partial computation of $\mathrm{Ann}_k$. The non-vanishing usually involves the explicit construction of non-zero classes. To this end one can use a duality property of $L^p$-cohomology (see [GT98, GT10, Pan08]): A closed differential $k$-form $\omega$ represents a non-zero class in $L^pH^k(M)$ (for $p > 1$) if, and only if, there exists a sequence of $L^q$-integrable differential $(n-k)$-forms $\{\beta_j\}_{j\in\mathbb{N}}$, with $n = \dim(M)$ and $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$\int_M \omega \wedge \beta_j \geq 1 \text{ and } \|d\beta_j\|_{L^q} \to 0.$$

In some cases (the real hyperbolic space for example) this construction is easy (see Section 2.4), but there are some problems to extend the techniques for more complicated spaces. For example, we can try to repeat the construction in more general Heintze groups of the form $\mathbb{R}^n \rtimes_\alpha \mathbb{R}$, but it requires to find a sequence of $L^q$-integrable $(n-k)$-forms $\{\beta_j\}_{j\in\mathbb{N}}$ with a determined value in the support of the $k$-form $\omega$. Here the first Sobolev inequality puts a restriction on the existence of such sequence. At this point the idea of relative $L^p$-cohomology appears, because it is the behaviour of the forms on a neighborhood of a special point on the boundary at infinity which hinders the construction of non-zero classes. This idea is to consider only the forms that vanish on a neighborhood of a fixed boundary point, which allows to avoid the problem described above.

It is important to say that the construction of non-zero classes using duality for Heintze groups as above can be done, but in a more sophisticated way than described. In [Pan08] there is a construction of a special class of non-zero cohomology classes called *torsion classes*.

A different idea of relative $L^p$-cohomology is defined in [BK12]. It is done in the simplicial case with respect to subcomplexes instead of boundary points.

Instead of the $L^p$-norm one can consider other Luxembourg norms defined by different Young functions. This gives the Orlicz cohomology, which is studied in recent works (see [Car16, GK19, Kop17, KP15]). This allows one to obtain finer quasi-isometry invariants. In particular, in [Car16] there are interesting results about the large scale geometry of Heinze groups obtained by using Orlicz cohomology in degree one. We expect that the study of higher degrees can give us new results too. To this end it is necessary to prove some fundamental properties of Orlicz cohomology such as equivalence theorems. We address these problems on Chapter 4.

## 1.2 Main definitions and results

### 1.2.1 $L^p$-cohomology

Let us consider $X$ a simplicial complex with finite dimension and a length distance $|\cdot - \cdot|$. Assume that there exist a constant $C \geq 0$ and a function $N : [0, +\infty) \to \mathbb{N}$ such that

(a) all simplices in $X$ have diameter smaller than $C$; and

(b) every ball with radius $r$ intersects at most $N(r)$ simplices.

In this case we say that $X$ has *bounded geometry*.

Fix a real number $p \in [1, +\infty)$ and consider for each $k \in \mathbb{N}$ the Banach space

$$\ell^p C^k(X) = \left\{ \theta : X^k \to \mathbb{R} : \sum_{\sigma \in X^k} |\theta(\sigma)|^p < +\infty \right\}$$

with the usual $\ell^p$-norm

$$\|\theta\|_{\ell^p} = \left( \sum_{\sigma \in X^k} |\theta|^p \right)^{\frac{1}{p}},$$

where $X^k$ denotes the set of $k$-simplices in $X$. The coboundary operator

$$\delta = \delta_k : \ell^p C^k(X) \to \ell^p C^{k+1}(X)$$

is defined by $\delta_k(\theta)(\sigma) = \theta(\partial\sigma)$, where $\partial$ is the usual boundary operator. It is easy to see, using bounded geometry, that $\delta_k$ is continuous (see Section 3.1). The *$k$-space of $\ell^p$-cohomology of $X$* is the topological vector space

$$\ell^p H^k(X) = \frac{\operatorname{Ker}\delta_k}{\operatorname{Im}\delta_{k-1}}.$$

It is sometimes convenient to consider also the *$k$-spaces of reduced $\ell^p$-cohomology of $X$* as the Banach space

$$\ell^p \overline{H}^k(X) = \frac{\operatorname{Ker}\delta_k}{\overline{\operatorname{Im}\delta_{k-1}}}.$$

Let us assume now that $X$ is Gromov-hyperbolic. For a point $\xi \in \partial X$ we denote by $\ell^p C^k(X, \xi)$ the subspace of $\ell^p C^k(X)$ consisting of all $k$-cochains that are zero on a neighborhood of $\xi$ in $\overline{X}$. We say that a $k$-cochain $\theta$ is zero or vanishes on $U \subset \overline{X}$ if for every $k$-simplex $\sigma \subset U$ we have $\theta(\sigma) = 0$. Note that $\ell^p C^k(X, \xi)$ is not a closed subspace, so it is not a Banach space.

The coboundary operator $\delta_k$ maps $\ell^p C^k(X, \xi)$ on $\ell^p C^{k+1}(X, \xi)$; hence for every $k \in \mathbb{N}$ and $p \in [1, +\infty)$ we define the *$k$-space of $\ell^p$-cohomology of $X$ relative to $\xi$* as the quotient

$$\ell^p H^k(X, \xi) = \frac{\operatorname{Ker}\delta|_{\ell^p C^k(X,\xi)}}{\operatorname{Im}\delta|_{\ell^p C^{k-1}(X,\xi)}}.$$

In Section 2.2 we prove the following result:

**Theorem 1.2.1.** *Let $X$ and $Y$ be two Gromov-hyperbolic and uniformly contractible simplicial complexes with finite dimension and bounded geometry, and $\xi$ a fixed point in $\partial X$. If $F : X \to Y$ is a quasi-isometry, then for every $p \in [1, +\infty)$ and $k \in \mathbb{N}$ there is an isomorphism of topological vector spaces between $\ell^p H^k(X, \xi)$ and $\ell^p H^k(Y, F(\xi))$.*

A metric space $X$ is *uniformly contractible* if it is contractible and there is a function $\psi : [0, +\infty) \to [0, +\infty)$ such that every ball $B(x, r) = \{x' \in X : |x' - x| < r\}$ is contractible into the ball $B(x, \psi(r))$.

Theorem 1.2.1 is also true for $\ell^p$-cohomology in the classical sense, see [Gro93, BP03]. In fact, the proof we give in Section 2.2 is an adaptation of the proof of Theorem 1.1 in [BP03].

In order to define the de Rham version of $L^p$-cohomology consider a Riemannian manifold $M$ of dimension $n$, an integer $k = 0, \dots, n$ and $p \in [1, +\infty)$. Let us set some definitions and notations:

(i) Denote $\Lambda^k(M) = \bigcup_{x\in M} \Lambda^k(T_xM)$, where $\Lambda^k(T_xM)$ is the space of alternating $k$-linear maps on the tangent space $T_xM$. A *$k$-form* on $M$ is a function $\omega : M \to \Lambda^k(M)$, $x \mapsto \omega_x$, satisfying $\omega_x \in \Lambda^k(T_xM)$ for all $x \in M$.

(ii) If $\psi : U \subset \mathbb{R}^n \to M$ is a parametrization, then we can write the pull-back of a $k$-form $\omega$ on $M$, defined by $\psi^*\omega_x(v_1,\dots,v_k) = \omega_{\psi(x)}(d_x\psi(v_1),\dots,d_x\psi(v_1))$, as

$$\psi^*\omega = \sum_{1\leq i_1<\dots<i_k\leq n} a_{i_1\dots i_k}\, dx_{i_1}\wedge\dots\wedge dx_{i_k},$$

where the real functions $a_{i_1\dots i_k} : U \to \mathbb{R}$ are called *coefficients of $\omega$ with respect to the parametrization $\psi$*. We say that $\omega$ is *measurable* if the coefficients of $\omega$ with respect to every parametrization are Borel measurable. It is said to be *smooth* or a *differential $k$-form* if the coefficients are smooth. The space of differential $k$-forms on $M$ is denoted by $\Omega^k(M)$.

(iii) The *operator norm* of a $k$-form $\omega$ in $M$ at the point $x$ is

$$|\omega|_x = \sup\{|\omega_x(v_1,\dots,v_k)| : v_i \in T_xM \text{ for } i = 1,\dots,k, \text{ with } \|v_i\|_x = 1\},$$

where $\|\ \|_x$ is the Riemannian norm in $T_xM$. We say that a $k$-form is *$L^p$-integrable* (resp. *$L^p$-locally integrable*) if it is measurable and the function $x \mapsto |\omega|_x$ is in $L^p(M)$ (resp. $L^{p,loc}(M)$). In the case $p = 1$ we just say that $\omega$ is *integrable* (resp. *locally integrable*). We denote by $L^p(M,\Lambda^k)$ the space of $L^p$-integrable $k$-forms on $M$ up to almost everywhere zero forms, which is a Banach space equipped with the $L^p$-norm

$$\|\omega\|_{L^p} = \left(\int_M |\omega|_x^p dV(x)\right)^{\frac{1}{p}},$$

where $dV$ is the Riemannian volume on $M$.

Consider the space

$$L^p\Omega^k(M) = \{\omega \in \Omega^k(M) : \|\omega\|_{L^p}, \|d\omega\|_{L^p} < +\infty\}.$$

It is not complete with the norm $|\omega|_{L^p} = \|\omega\|_{L^p} + \|d\omega\|_{L^p}$, so we consider its completion $L^pC^k(M)$. Observe that the usual derivative is continuous in $(L^p\Omega^k(M), |\ \ |_{L^p})$, thus it can be extended to a continuous function $d = d_k : L^pC^k(M) \to L^pC^{k+1}(M)$.

The *$k$-space of $L^p$-cohomology of $M$* is

$$L^pH^k(M) = \frac{\operatorname{Ker} d_k}{\operatorname{Im} d_{k-1}}.$$

As before, we have also the *$k$-space of reduced $L^p$-cohomology* as the Banach space

$$L^p\overline{H}^k(M) = \frac{\operatorname{Ker} d_k}{\overline{\operatorname{Im} d_{k-1}}}.$$

**Remark 1.2.2.** If $\omega$ is a locally integrable $k$-form in $M$ we say that another locally integrable $(k+1)$-form $\varpi$ is its *weak derivative* and we write $\varpi = d\omega$ if for every $\beta \in \Omega^{n-k-1}(M)$ with compact support

$$\int_M \varpi \wedge \beta = (-1)^{k+1} \int_M \omega \wedge d\beta.$$

We say that $\omega$ is *closed* if it have the $(k+1)$-form constant zero as its weak derivative, and that it is *exact* if there exists a locally integrable $(k-1)$-form $\vartheta$ such that $d\vartheta = \omega$. Then an equivalent definition of de Rham $L^p$-cohomology can be done considering the quotient $Z^{k,p}(M)/B^{p,k}(M)$, where $Z^{k,p}(M)$ is the space of closed $k$-forms and $B^{k,p}(M)$ is the space of exact $k$-forms in $L^p(M, \Lambda^k)$.

Since $L^p(M, \Lambda^k)$ is complete and contains $L^p\Omega^k(M)$, every form in $L^pC^k(M)$ can be seen as an element of $L^p(M, \Lambda^k)$.

Using Hölder's inequality (see Lemma 2.3.3) we can see that every $k$-form in $L^pC^k(M)$ has weak derivative in $L^pC^{k+1}(M)$, thus the equivalence between both definitions of de Rham $L^p$-cohomology follows from [GT10, Proposition 2], whose proof is based on regularisation methods (see for example [GKS84, GT06]).

If $M$ is complete and Gromov-hyperbolic and $\xi$ is a point in $\partial M$, we can consider again $L^pC^k(M, \xi)$ the subspace of $L^pC^k(M)$ consisting of all $k$-forms that vanish (almost everywhere) on a neighborhood of $\xi$ in $\overline{M}$. The *$k$-space of $L^p$-cohomology of $M$ relative to $\xi$* is

$$L^pH^k(M, \xi) = \frac{\operatorname{Ker} d|_{L^pC^k(M,\xi)}}{\operatorname{Im} d|_{L^pC^{k-1}(M,\xi)}}.$$

Given such a pair $(M, \xi)$, where $M$ has bounded geometry, there exists a pair $(X_M, \overline{\xi})$ called *a simplicial pair associated to* $(M, \xi)$, where $X_M$ is a simplicial complex with finite dimension and bounded geometry that is quasi-isometric to $M$, and $\overline{\xi} \in \partial X_M$ corresponds to $\xi$ by the quasi-isometry between $M$ and $X_M$. The simplicial complex $X_M$ will be constructed as a nerve of a covering (see the precise definition in Section 3.2). Then we have the following result:

**Theorem 1.2.3.** *Let $M$ be a complete and Gromov-hyperbolic Riemannian manifold with bounded geometry and $\xi \in \partial M$. Take $(X_M, \overline{\xi})$ a simplicial pair associated to $(M, \xi)$. Then for all $p \in [1, +\infty)$ and $k \in \mathbb{N}$ there is a canonical isomorphism between $L^pH^k(M, \xi)$ and $\ell^pH^k(X_M, \overline{\xi})$.*

The proof of this result is done in Section 3.2. It is again an adaptation of the proof in the classical case (see [Pan95, Gen14]).

From the proof of Theorem 1.2.3 it follows:

**Theorem 1.2.4.** *If $M$ is a complete and Gromov-hyperbolic Riemannian manifold with bounded geometry, then for every point $\xi \in \partial M$ and $p \in [1, +\infty)$ the cochain complexes $(L^pC^*(M, \xi), d)$ and $(L^p\Omega^*(M, \xi), d)$ are homotopically equivalent.*

If $M$ is uniformly contractible, then so is $X_M$; therefore Theorem 1.2.3 implies:

**Corollary 1.2.5.** *Let $F : M \to N$ be a quasi-isometry between two complete, uniformly contractible and Gromov-hyperbolic Riemannian manifolds with bounded geometry. Then for every point $\xi$ in $\partial M$ the spaces $L^pH^k(M,\xi)$ and $L^pH^k(N,F(\xi))$ are isomorphic for all $p \in [1,+\infty)$ and $k \in \mathbb{N}$.*

### 1.2.2 Heintze groups

A result by Heintze ([Htz74]) says that every homogeneous and connected Riemannian manifold with negative sectional curvature is isometric to a Lie group of the form $N \rtimes_\tau \mathbb{R}$ with a left-invariant Riemannian metric. Here $N$ is a connected and simply connected nilpotent Lie group and the homomorphism $\tau : \mathbb{R} \to \mathrm{Aut}(N)$ satisfies $d_e\tau(t) = e^{t\alpha}$, where $\alpha$ is a derivation on the Lie algebra $\mathrm{Lie}(N)$ with all its eigenvalues with positive real part. Moreover, if $N \rtimes_\tau \mathbb{R}$ is such a group, then there exists a left-invariant Riemannian metric in $N \rtimes_\tau \mathbb{R}$ with negative sectional curvature. A group with this structure is called a *Heintze group* and will be denoted by $N \rtimes_\alpha \mathbb{R}$ if $\tau$ is determined by $\alpha$.

We are interested in the restriction of Problem 1.1.1 to the family of Heintze groups. About this we first observe that two left-invariant metrics on a Lie group are always bi-Lipschitz equivalent, thus the quasi-isometry class of a Lie group does not depend on the choice of the left-invariant metric. In particular a Heintze group with any left-invariant metric is always hyperbolic in the sense of Gromov. This also shows that two isomorphic Heintze groups are quasi-isometric. The converse is not true in general: every Heintze group is quasi-isometric to a *purely real* Heintze group, which is determined by a derivation with real eigenvalues (see [Cor18]). If we restrict the problem to purely real Heintze groups we have the following conjecture:

**Conjecture 1.2.6** ([Cor18])**.** *Two purely real Heintze groups are quasi-isometric if, and only if, they are isomorphic.*

This conjecture remains open in its full generality, however, there are some partial results. For instance, this is proved in the case of Heintze groups of Carnot type ([Pan89]) and for groups with the form $\mathbb{R}^n \rtimes_\alpha \mathbb{R}$ ([Xie14]). See also [Pan08, SX12, Xie15a, Xie15b, CS17] for related results and particular cases.

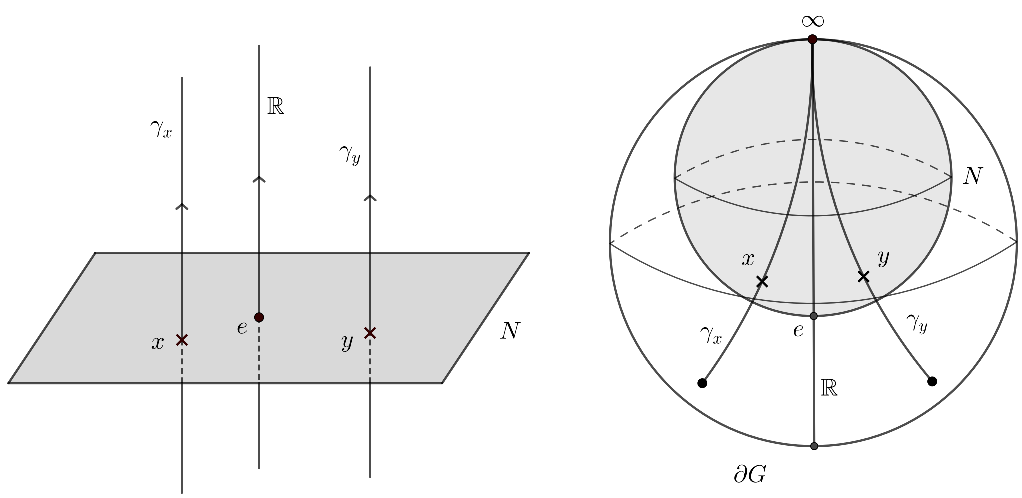
We are interested in finding quasi-isometry invariants related to $L^p$-cohomology. Using the relative $L^p$-cohomology we obtain a proof of the following result (Section 3.4):

**Theorem 1.2.7.** *Let $G_1 = \mathbb{R}^{n-1} \rtimes_{\alpha_1} \mathbb{R}$ and $G_2 = \mathbb{R}^{n-1} \rtimes_{\alpha_2} \mathbb{R}$ be two purely real Heintze groups. If $G_1$ and $G_2$ are quasi-isometric, then there exists $\lambda > 0$ such that $\alpha_1$ and $\lambda\alpha_2$ have the same eigenvalues counted with multiplicity.*

Observe that the Heintze groups $\mathbb{R}^{n-1} \rtimes_\alpha \mathbb{R}$ and $\mathbb{R}^{n-1} \rtimes_{\lambda\alpha} \mathbb{R}$ are isomorphic if $\lambda > 0$. Therefore the invariance of the eigenvalues is not true.

The original proof of Theorem 1.2.7 is done in [Pan08]. The strategy used by Pansu is to compute the sets $\mathrm{Ann}_k(\mathbb{R}^{n-1} \rtimes_\alpha \mathbb{R})$ for $k \geq 2$. From this he obtains some critical exponents related to the eigenvalues of $\alpha$, which are invariant by quasi-isometry. The difficult part of this proof is to construct non-zero cohomology classes for the exponents where the $L^p$-cohomology is not zero. The advantage of the relative version is that the construction of these non-zero classes is easier than in the classical case. A proof of Theorem 1.2.7 that uses different methods can be found in [Xie14]. In general, it is proved in [CS17] that Theorem 1.2.7 is true for every pair of quasi-isometric purely real Heintze groups $G_1$ and $G_2$. This is done using an induction argument and results given in [Pan89, Xie14, LX16, Car16].

There is an important fact about the boundary of a Heintze group $G = N \rtimes_\alpha \mathbb{R}$: All vertical lines $\gamma_x(t) = (x, t)$, for $x \in N$, are geodesics asymptotic to the future, they determine a special point in $\partial G$ denoted by $\infty$. One can also prove that all points in $\partial G \setminus \{\infty\}$ are represented by an unique vertical line (to the past), so we can write the boundary as $\partial G = N \cup \{\infty\}$.




### 1.2.3 Orlicz cohomology

We say that a real function $\phi : \mathbb{R} \to [0, +\infty)$ is a *Young function* if:

- is even and convex;
- $\phi(t) = 0$ if, and only if, $t = 0$.

Observe that every Young function $\phi$ satisfies

$$\lim_{t\to+\infty}\phi(t)=+\infty.$$

We say that $\phi$ is *doubling* if there exists a constant $D\geq 2$ such that $\phi(2t)\leq D\phi(t)$ for all $t>0$.

Let $(Z,\mu)$ be a measure space and $\phi$ a Young function. The *Luxembourg norm associated to* $\phi$ of a measurable function $f:Z\to\overline{\mathbb{R}}=[-\infty,+\infty]$ is defined by

$$\|f\|_{L^\phi}=\inf\left\{\gamma>0:\int_Z\phi\left(\frac{f}{\gamma}\right)d\mu\leq 1\right\}\in[0,+\infty].$$

The *Orlicz space of* $(Z,\mu)$ *associated to* $\phi$ is the Banach space

$$L^\phi(Z,\mu)=\frac{\{f:Z\to\overline{\mathbb{R}}\text{ measurable}:\|f\|_{L^\phi}<+\infty\}}{\{f:Z\to\overline{\mathbb{R}}\text{ measurable}:\|f\|_{L^\phi}=0\}}.$$

It is not difficult to see that $\|f\|_{L^\phi}=0$ if, and only if, $f=0$ almost everywhere.

If $\mu$ is the counting measure on $Z$ we denote $L^\phi(Z,\mu)=\ell^\phi(Z)$ and $\|\ \|_{L^\phi}=\|\ \|_{\ell^\phi}$. Observe that if $\phi$ is the function $t\mapsto|t|^p$, then $L^\phi(Z,\mu)$ is the classic space $L^p(Z,\mu)$. We refer to [RR91] for a background about Orlicz spaces.

We can define the simplicial $\ell^\phi$-cohomology in the same way we defined the $\ell^p$-cohomology, i.e, for a finite dimensional simplicial complex $X$ with bounded geometry we set $\ell^\phi C^k(X)=\ell^\phi(X^k)$. The coboundary operator $\delta_k:\ell^\phi C^k(X)\to\ell^\phi C^{k+1}(X)$ is continuous (see Section 4.1), then we define respectively the *$k$-space of $\ell^\phi$-cohomology* and *reduced $\ell^\phi$-cohomology of* $X$ as

$$\ell^\phi H^k(X)=\frac{\operatorname{Ker}\delta_k}{\operatorname{Im}\delta_{k-1}}\quad\text{and}\quad\ell^\phi\overline{H}^k(X)=\frac{\operatorname{Ker}\delta_k}{\overline{\operatorname{Im}\delta_{k-1}}}.$$

As before, if $X$ is Gromov-hyperbolic and $\xi\in\partial X$ we can consider the *$k$-space* of *$\ell^\phi$-cohomology relative to* $\xi$ as

$$\ell^\phi H^k(X,\xi)=\frac{\operatorname{Ker}\delta|_{\ell^\phi C^k(X,\xi)}}{\operatorname{Im}\delta|_{\ell^\phi C^{k-1}(X,\xi)}},$$

where $\ell^\phi C^k(X,\xi)$ is the subspace of $\ell^\phi C^k(X)$ consisting of all $k$-cochains which are zero on a neighborhood of $\xi$ in $\overline{X}$.

A $\ell^\phi$-version of Theorem 1.2.1 holds:

**Theorem 1.2.8.** *Let $X$ and $Y$ be two uniformly contractible Gromov-hyperbolic simplicial complexes with finite dimension and bounded geometry, and $\phi$ a Young function. If $F:X\to Y$ and $\xi\in\partial X$, then for every $k\in\mathbb{N}$ there is an isomorphism between $\ell^\phi H^k(X,\xi)$ and $\ell^\phi H^k(Y,F(\xi))$.*

We prove Theorem 1.2.8 in Section 4.1. The proof of the non-relative version of the theorem can be read in [Car16].

If $M$ is a Riemannian manifold, consider

$$L^\phi\Omega^k(M) = \{\omega \in \Omega^k(M) : \|\omega\|_{L^\phi}, \|d\omega\|_{L^\phi} < +\infty\},$$

equipped with the norm $|\omega|_{L^\phi} = \|\omega\|_{L^\phi} + \|d\omega\|_{L^\phi}$, and $L^\phi C^k(M)$ its completion. The derivative $d_k : L^\phi C^k(M) \to L^\phi C^{k+1}(M)$ is continuous, then we define the *$k$-space of $L^\phi$-cohomology of $M$* as

$$L^\phi H^k(M) = \frac{\operatorname{Ker} d_k}{\operatorname{Im} d_{k-1}},$$

and its *$k$-space of reduced $L^\phi$-cohomology* as

$$L^\phi \overline{H}^k(M) = \frac{\operatorname{Ker} d_k}{\overline{\operatorname{Im} d_{k-1}}}.$$

We also call this family of spaces *the Orlicz-de Rham cohomology of $M$ associated to the Young function $\phi$.*

As in the $L^p$-case, we can also consider $L^\phi(M, \Lambda^k)$ the space of $L^\phi$-integrable $k$-forms up to almost everywhere zero forms. We can see the elements of $L^\phi C^k(M)$ as $k$-forms in $L^\phi(M, \Lambda^k)$.

If $M$ is Gromov-hyperbolic and $\xi \in \partial M$, then we can define the *$k$-space of relative Orlicz-de Rham cohomology of the pair $(M, \xi)$ for the Young function $\phi$* (or *$k$-space of $L^\phi$-cohomology of $M$ relative to $\xi$*) as

$$L^\phi H^k(M, \xi) = \frac{\operatorname{Ker} d|_{L^\phi C^k(M,\xi)}}{\operatorname{Im} d|_{L^\phi C^{k-1}(M,\xi)}},$$

where $L^\phi C^k(M, \xi)$ denotes the subspace of $k$-forms in $L^\phi C^k(M)$ which vanish on some neighborhood of $\xi$.

As we see in Section 4.2 the generalization of Theorem 1.2.3 presents some difficulties. A proof in the case of degree one can be found in [Car16, Section 3]. We give a proof in the case of Lie groups.

**Theorem 1.2.9.** *Let $G$ be a Lie group equipped with a left-invariant Riemannian metric and $X_G$ the corresponding simplicial complex as in Section 1.2.1. Consider $\phi$ a doubling Young function. Then the cochain complexes $(\ell^\phi C^k(X_G), \delta)$, $(L^\phi C^k(G), d)$ and $(L^\phi \Omega^k(G), d)$ are homotopically equivalent. Moreover, if $G$ is Gromov-hyperbolic and $\xi$ is a point in $\partial G$, then the cochain complexes $(\ell^\phi C^k(X_G, \bar{\xi}), \delta)$, $(L^\phi C^k(G, \xi), d)$ and $(L^\phi \Omega^k(G, \xi), d)$ are homotopically equivalent, where $(X_G, \overline{\xi})$ is a simplicial pair associated to $(M, \xi)$. As a consequence the corresponding cohomology spaces are isomorphic.*

A consequence of the previous theorem is the quasi-isometry invariance of Orlicz-de Rham cohomology in the case of Lie groups.

**Corollary 1.2.10.** *If $F : G_1 \to G_2$ is a quasi-isometry between two contractible Lie groups equipped with left-invariant metrics and $\phi$ is a doubling Young function, then for every $k \in \mathbb{N}$ the topological vector spaces $L^\phi H^k(G_1)$ and $L^\phi H^k(G_2)$ are isomorphic. Furthermore, if $G_1$ and $G_2$ are Gromov-hyperbolic and $\xi$ is a point in $\partial G_1$, then the spaces $L^\phi H^k(G_1, \xi)$ and $L^\phi H^k(G_2, F(\xi))$ are isomorphic for every $k$.*

As we said in Section 1.1, a motivation to study Orlicz cohomology is to find finer quasi-isometry invariants related to Heintze groups. Considering a larger family of Young functions could improve Theorem 1.2.7. It is interesting for example consider the following question:

**Question 1.2.11.** *Let $N_1 \rtimes_{\alpha_1} \mathbb{R}$ and $N_2 \rtimes_{\alpha_2} \mathbb{R}$ be two quasi-isometric purely real Heintze groups. Is there a positive number $\lambda > 0$ such that $\alpha_1$ and $\lambda\alpha_2$ have the same Jordan form?*

As an example of application of Orlicz cohomology to the Problem 1.1.1 and in particular to Question 1.2.11 for Heintze groups we can consider the family of doubling Young functions given by

$$\phi_{p,\kappa}(t) = \frac{|t|^p}{\log(e + |t|^{-1})^\kappa},$$

with $p \in [1, +\infty)$ and $\kappa \in [0, +\infty)$. We put the lexicographic order in the family of indices $(p, \kappa)$ and denote $L^{p,\kappa} H^k(G) = L^{\phi_{p,\kappa}} H^k(G)$.

For degree one consider the critical exponent

$$p_{\neq 0}(G) = \inf\{(p, \kappa) \in [1, +\infty) \times [0, \infty) : L^{p,\kappa} H^1(G) \neq 0\}.$$

Then we have the following result:

**Theorem 1.2.12** ([Car16])**.** *Let $G = N \rtimes_\alpha \mathbb{R}$ be a purely real Heintze group where $\alpha$ has eigenvalues $\lambda_1 < \ldots < \lambda_d$. Then*

$$p_{\neq 0}(G) = \left( \frac{\operatorname{tr}(\alpha)}{\lambda_1}, 1 + \frac{\operatorname{tr}(\alpha)}{\lambda_1}(m_1 - 1) \right),$$

*where $m_1$ is the size of the biggest Jordan subblock associated to $\lambda_1$.*

As a conclusion of Theorem 1.2.12 one obtain that $m_1$ is invariant under quasi-isometries between Heintze groups, which improves Theorem 1.2.7 and gives us a partial answer to the Question 1.2.11. This motivates the study of Orlicz cohomology in higher degrees.

Other results related to Question 1.2.11 and the large scale geometry of Heintze groups in general are obtained using a local version of Orlicz geometry in [Car16].

# Chapter 2

# Preliminaries

The aim of this chapter is to set notation and state some lemmas that we will use in the following chapters.

## 2.1 Quasi-isometries and Gromov-hyperbolic spaces

Let $X$ and $Y$ be two metric spaces, we denote the distance by $|\cdot - \cdot|$ in both cases. A map $F : X \to Y$ is a *quasi-isometry embedding* if there exist two constants $\lambda \geq 1$ and $\epsilon \geq 0$ such that for all $x, x' \in X$,

$$\lambda^{-1}|x - x'| - \epsilon \leq |F(x) - F(x')| \leq \lambda|x - x'| + \epsilon.$$

We say that $F$ is a *quasi-isometry* if we also have that $F(X)$ is $C$-dense in $Y$ for some $C \geq 0$, which means that for every $y \in Y$ there exists $x \in X$ such that $|F(x) - y| \leq C$. In this case we say that $X$ and $Y$ are *quasi-isomtric spaces*.

**Remark 2.1.1.** Consider $F : X \to Y$ and $G : Y \to Z$ two quasi-isometries. It is easy to see that:

(i) $G \circ F : X \to Z$ is a quasi-isometry.

(ii) There exists a quasi-isometry $\overline{F} : Y \to X$ such that $F \circ \overline{F}$ and $\overline{F} \circ F$ are at bounded uniform distance from the identity. We say that $\overline{F}$ is a *quasi-inverse* of $F$.

These two conditions give us an equivalence relation between metric spaces. In this context Problem 1.1.1 appears naturally.

There is a natural relation between quasi-isometries: If $F, G : X \to Y$ are two quasi-isometries, we write $F \sim G$ if the uniform distance between $F$ and $G$ is bounded.

Under this equivalence the quasi-inverse of a quasi-isometry is unique, then we can consider the *group of quasi-isometries* of some metric space $X$ as

$$QI(X) = \{F : X \to X : F \text{ is a quasi-isometry}\}/\sim .$$

Observe that the composition of quasi-isometries passes to the quotient; hence it defines a product on $QI(X)$. We also use the notation

$$QI(X,Y) = \{F : X \to Y : F \text{ is a quasi-isometry}\}/\sim .$$

Here we find another general problem linked to quasi-isometries:

**Problem 2.1.2.** *Given a metric space $X$, how does $QI(X)$ act on it?*

A geodesic metric space $X$ is *Gromov-hyperbolic* if there exists $\delta > 0$ such that every geodesic triangle $\Delta = [x,y] \cup [x,z] \cup [y,z]$ is contained in a $\delta$-neighborhood of any two of its edges. In this case we say also that $X$ is *$\delta$-hyperbolic in the sense of Gromov*.

The proof of the following theorem can be found in [GdlH90, Chapter 5].

**Theorem 2.1.3.** *Let $X$ and $Y$ be two geodesic metric spaces and $F : X \to Y$ a quasi-isometry. If $Y$ is Gromov-hyperbolic, then so is $X$.*

There is a general definition of Gromov-hyperbolic spaces for non-geodesic spaces. We will not give it, but the reader can find it in [GdlH90, Chapter 2].

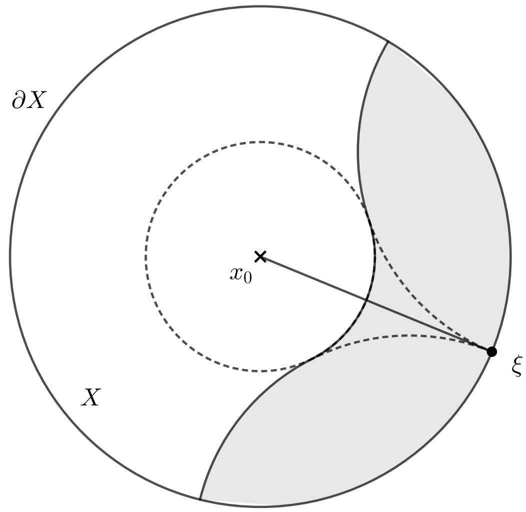
The *boundary at infinity* (or simply *boundary*) of a geodesic and proper Gromov-hyperbolic metric space $X$ is defined as the set of equivalence classes of geodesic rays in $X$ up to bounded Hausdorff distance. We denote it by $\partial X$.

The set $\overline{X} = X \cup \partial X$ has a natural topology for which it is a compactification of $X$. Indeed, it can be seen as the topology induced by a metric $d$ on $\overline{X}$ such that for all $x, x' \in \overline{X}$,

$$d(x,x') \asymp a^D \min\{1, |x - x'|\},\ D = dist(x_0, [x,x']), \tag{2.1}$$

where $a > 0$ and $x_0 \in X$ are fixed, and $[x,x']$ denotes a geodesic (or a geodesic ray or a geodesic segment) between $x$ and $x'$. If one of these points belongs to the boundary we put $|x - x'| = +\infty$. See for example [BHK01, Chapter 4] for more details. If $f$ and $g$ are two real functions with the same domain we write $f \asymp g$ if there exists a uniform constant $C \geq 1$ such that $C^{-1} f \leq g \leq Cf$.

The following figure shows such a neighborhood of a point $\xi \in \partial X$ in $\overline{X}$, which approximates a ball with centre $\xi$ for the distance (2.1).




If $F : X \to Y$ is a quasi-isometry between two Gromov-hyperbolic metric spaces, then it induces a homeomorphism between their boundaries $\partial F : \partial X \to \partial Y$ (see [GdlH90, Chapter 7,Section 4]). In order to simplify the notation we will also write $F(\xi) = \partial F(\xi)$ if $\xi$ is a point in $\partial X$. It follows directly from the construction of the boundary map that if $G \sim F$, then $\partial F = \partial G$.

## 2.2 Continuous cochain complexes

A *continuous cochain complex* is a sequence of topological vector spaces $\{V^k\}_{k\in\mathbb{Z}}$ and continuous linear maps $\{d_k\}_{k\in\mathbb{Z}}$ as in the following diagram such that $\operatorname{Im} d_{k-1} \subset \operatorname{Ker} d_k$ for all $k \in \mathbb{Z}$. We denote it by $(V^*, d_*)$ or $(V^*, d)$.

$$\cdots \xrightarrow{d_{-3}} V^{-2} \xrightarrow{d_{-2}} V^{-1} \xrightarrow{d_{-1}} V^0 \xrightarrow{d_0} V^1 \xrightarrow{d_1} V^2 \xrightarrow{d_2} \cdots$$

The *k-space of cohomology* of the continuous cochain complex $(V^*, d)$ is the quotient $\operatorname{Ker} d_k / \operatorname{Im} d_{k-1}$. This is a topological vector space with the quotient topology. We can consider also the *k-space of reduced cohomology* as $\operatorname{Ker} d_k / \overline{\operatorname{Im} d_{k-1}}$. In order to simplify the notation, we will say *cochain complex* to refer to a continuous cochain complex.

A cochain map between two cochain complexes $(V^*, d)$ and $(W^*, \delta)$ is a family of continuous linear maps $f_k : V^k \to W^k$ such that $\delta_k \circ f_k = f_{k+1} \circ d_k$. In general we write $f = f_k$ if it is not necessary to specify its degree. A cochain map induces naturally a continuous linear map between the corresponding (reduced) cohomology spaces.

Two cochain maps $f$ and $g$ from $(V^*, d)$ to $(W^*, \delta)$ are *homotopic* if there is a family of continuous linear maps $h = h_k : V^k \to W^{k-1}$ such that $h_{k+1} \circ d_k + \delta_{k-1} \circ h_k = f_k - g_k$ for every $k \in \mathbb{Z}$. In this case we say that $h$ is an *homotopy* between $f$ and $g$. If $f$ and $g$ are homotopic, then they induce the same map in (reduced) cohomology.

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & V^{-2} \arrow[r, "d"] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & V^{-1} \arrow[r, "d"] \arrow[dl, "h"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & V^{0} \arrow[r, "d"] \arrow[dl, "h"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & V^{1} \arrow[r, "d"] \arrow[dl, "h"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & V^{2} \arrow[r, "d"] \arrow[dl, "h"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & \cdots \\
\cdots \arrow[r, "\delta"] & W^{-2} \arrow[r, "\delta"] & W^{-1} \arrow[r, "\delta"] & W^{0} \arrow[r, "\delta"] & W^{1} \arrow[r, "\delta"] & W^{2} \arrow[r, "\delta"] & \cdots
\end{tikzcd}$$

We say that two cochain complexes are *homotopically equivalent* if there exist cochain maps $f : (V^*, d) \to (W^*, \delta)$ and $g : (W^*, \delta) \to (V^*, d)$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity. In this case $f$ and $g$ induce isomorphisms of topological vector spaces between the corresponding (reduced) cohomology spaces. This defines an equivalence relation between cochain complexes.

A cochain complex $(V^*, d)$ *retracts* to a subcomplex $(U^*, d)$ ($U^k \subset V^k$ for all $k \in \mathbb{Z}$) if there exists a cochain map $r : (V^*, d) \to (U^*, d)$ such that $i \circ r$ is homotopic to the identity, where $i$ is the inclusion, and the homotopy $h$ satisfies $h(U^k) \subset U^{k-1}$ for every $k \in \mathbb{N}$. This implies that $(V^*, d)$ and $(U^*, d)$ are homotopically equivalent.

By a *bicomplex* we mean a family of topological vector spaces $\{C^{k,\ell}\}_{(k,\ell)\in\mathbb{N}^2}$ and continuous maps $d' : C^{k,\ell} \to C^{k+1,\ell}$ and $d'' : C^{k,\ell} \to C^{k,\ell+1}$ such that all rows and columns $(C^{*,\ell}, d')$ and $(C^{k,*}, d'')$ are cochain complexes. We denote it by $(C^{*,*}, d', d'')$.

The following lemma will be important in Sections 3.2 and 4.2.

**Lemma 2.2.1** (Lemma 5,[Pan95])**.** *Let $(C^{*,*}, d', d'')$ be a bicomplex with $d' \circ d'' + d'' \circ d' = 0$. Suppose that for every $\ell \in \mathbb{N}$, the complex $(C^{*,\ell}, d')$ retracts to the subcomplex $(E^\ell := \operatorname{Ker} d'|_{C^{0,\ell}} \to 0 \to 0 \to \cdots)$. Then the complex $(D^*, \delta)$, defined by*

$$D^m = \bigoplus_{k+\ell=m} C^{k,\ell} \text{ and } \delta = d' + d'',$$

*is homotopically equivalent to $(E^*, d'')$.*

$$\begin{tikzcd}
 & & D^0 \arrow[r, "\delta"] \arrow[dl, dash, dotted] & D^1 \arrow[r, "\delta"] \arrow[dl, dash, dotted] & D^2 \arrow[dl, dash, dotted] \\
E^0 \arrow[r, hook] \arrow[d, "d''"] & C^{0,0} \arrow[r, "d'"] \arrow[d, "d''"] & C^{1,0} \arrow[r, "d'"] \arrow[d, "d''"] \arrow[dl, dash, dotted] & C^{2,0} \arrow[r, "d'"] \arrow[d, "d''"] \arrow[dl, dash, dotted] & \cdots \arrow[dl, dash, dotted] \\
E^1 \arrow[r, hook] \arrow[d, "d''"] & C^{0,1} \arrow[r, "d'"] \arrow[d, "d''"] & C^{1,1} \arrow[r, "d'"] \arrow[d, "d''"] \arrow[dl, dash, dotted] & C^{2,1} \arrow[r, "d'"] \arrow[d, "d''"] \arrow[dl, dash, dotted] & \cdots \arrow[dl, dash, dotted] \\
E^2 \arrow[r, hook] \arrow[d, "d''"] & C^{0,2} \arrow[r, "d'"] \arrow[d, "d''"] & C^{1,2} \arrow[r, "d'"] \arrow[d, "d''"] \arrow[dl, dash, dotted] & C^{2,2} \arrow[r, "d'"] \arrow[d, "d''"] \arrow[dl, dash, dotted] & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

*Proof.* For every $K \in \mathbb{N}$ let $(C^{*,*}_{[K]}, d', d'')$ be the subcomplex of $(C^{*,*}, d', d'')$ defined by

$$C^{k,\ell}_{[K]} = \begin{cases} C^{k,\ell} & \text{if } k < K \\ \operatorname{Ker} d'|_{C^{k,\ell}} & \text{if } k = K \\ 0 & \text{if } k > K \end{cases}.$$

Observe that it is a bicomplex because of the identity $d' \circ d'' + d'' \circ d' = 0$.

For every $m \in \mathbb{N}$ let

$$D^m_{[K]} = \bigoplus_{k+\ell=m} C^{k,\ell}_{[K]}.$$

One has $D^*_{[K]} \subset D^*_{[K+1]}$ for every $K$ and $\cup_{K \geq 0} D^*_{[K]} = D^*$. Moreover, by definition of $E^*$, one has $D^*_{[0]} = E^*$. Therefore, to prove the lemma, it will suffice to show that $D^*_{[K]}$ retracts to $D^*_{[K-1]}$ for every $K \geq 1$.

To construct the expected homotopies we first define some special maps denoted by $h'$ and $b$. In order to simplify the notation we set

$$\mathcal{C}_0 = \bigoplus_{\ell \geq 0} C^{0,\ell}, \quad \mathcal{C}_1 = \bigoplus_{k \geq 1, \ell \geq 0} C^{k,\ell}, \quad \text{and } \mathcal{E} = \bigoplus_{\ell \geq 0} E^{\ell}.$$

We also write $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$. By assumption, for every $\ell \in \mathbb{N}$, the complex $(C^{*,\ell}, d')$ retracts to the subcomplex $(E^{\ell} \to 0 \to 0 \to \cdots)$. Thus there exist continuous operators

$$h' : \mathcal{C}_1 \to \mathcal{C}, \text{ and } \varphi : \mathcal{C}_0 \to \mathcal{E},$$

such that

(i) $d' \circ h' + h' \circ d' = \mathrm{Id}$ on $\mathcal{C}_1$, and

(ii) $h' \circ d' = \mathrm{Id} - i \circ \varphi$ on $\mathcal{C}_0$,

where $i : \mathcal{E} \to \mathcal{C}_0$ is the inclusion. We extend $h'$ to the whole space $\mathcal{C}$ by letting $h' \equiv 0$ on $\mathcal{C}_0$.

Define $b : \mathcal{C} \to \mathcal{C}$ by

- $b = -(d'' \circ h' + h' \circ d'')$ on $\mathcal{C}_1$, and
- $b = i \circ \varphi$ on $\mathcal{C}_0$.

On the subspace $\mathcal{C}_1$ relation (i) implies that

$$\delta \circ h' + h' \circ \delta = \mathrm{Id} - b.$$

On $\mathcal{C}_0$ relation (ii) implies that

$$\delta \circ h' + h' \circ \delta = h' \circ d' = \mathrm{Id} - i \circ \varphi = \mathrm{Id} - b.$$

Therefore the relation $\delta \circ h' + h' \circ \delta = \mathrm{Id} - b$ is valid on the whole space $\mathcal{C}$. This implies in particular that $b$ commutes with $\delta$.

We are now ready to show that $D^*_{[K]}$ retracts to $D^*_{[K-1]}$ for every $K \geq 1$. Since $h'(C^{k,\ell}) \subset C^{k-1,\ell}$ for $k \geq 1$ and $h'(C^{0,\ell}) = 0$, we can consider $h'_{[K]} : D^m_{[K]} \to D^{m-1}_{[K]}$ the induced operator.

The map $b$ satisfies $b(C^{k,\ell}) \subset C^{k-1,\ell+1}$ for $k \geq 1$ and $b(C^{0,\ell}) \subset C^{0,\ell}$. Moreover for $K \geq 1$, one has

$$b(\operatorname{Ker} d'|_{C^{K,\ell}}) \subset \operatorname{Ker} d'|_{C^{K-1,\ell+1}}.$$

Indeed, if $d'\omega = 0$, then one has also $d'd''\omega = 0$. The definition of $b$ and the relation (i) yield :

$$d'b\omega = -(d'd''h'\omega + d'h'd''\omega) = d''d'h'\omega - d'h'd''\omega = d''\omega - d''\omega = 0.$$

Therefore $b$ sends every $D^m_{[K]}$ to $D^m_{[K-1]}$ for $K \geq 1$. Let $b_{[K]} : D^*_{[K]} \to D^*_{[K-1]}$ be the induced operator. As we saw above, it commutes with $\delta$. Since $\delta \circ h' + h' \circ \delta = \mathrm{Id} - b$ on the whole space $\mathcal{C}$, we get

$$\delta \circ h'_{[K]} + h'_{[K]} \circ \delta = \mathrm{Id} - i_{[K-1]} \circ b_{[K]}$$

and also

$$\delta \circ h'_{[K-1]} + h'_{[K-1]} \circ \delta = \mathrm{Id} - b_{[K]} \circ i_{[K-1]},$$

where $i_{[K-1]} : D^*_{[K-1]} \to D^*_{[K]}$ is the inclusion.

$$\begin{tikzcd}
D^0_{[K]} \arrow[r, "\delta"'] \arrow[d, bend right, "b"'] & D^1_{[K]} \arrow[r, "\delta"'] \arrow[l, bend right, "h'"'] \arrow[d, bend right, "b"'] & D^2_{[K]} \arrow[r, "\delta"'] \arrow[l, bend right, "h'"'] \arrow[d, bend right, "b"'] & D^3_{[K]} \arrow[r, "\delta"'] \arrow[l, bend right, "h'"'] \arrow[d, bend right, "b"'] & \cdots \\
D^0_{[K-1]} \arrow[r, "\delta"] \arrow[u, bend right, "i"'] & D^1_{[K-1]} \arrow[r, "\delta"] \arrow[l, bend left, "h'"] \arrow[u, bend right, "i"'] & D^2_{[K-1]} \arrow[r, "\delta"] \arrow[l, bend left, "h'"] \arrow[u, bend right, "i"'] & D^3_{[K-1]} \arrow[r, "\delta"] \arrow[l, bend left, "h'"] \arrow[u, bend right, "i"'] & \cdots
\end{tikzcd}$$

All maps in the diagram are continuous, then the lemma follows. $\square$

## 2.3 Some properties about integration of forms

Suppose that $M$ is a smooth manifold of dimension $n$ and $(Z, \mu)$ is a measure space. We say that a function $\Phi : M \times Z \to \Lambda^k(M)$ is a *measurable family of $k$-forms* on $M$ if for all $z \in Z$ the function $x \mapsto \Phi_{(x,z)}$ is a $k$-form on $M$ and the coefficients of $\Phi$ with respect to every parametrization (depending on $x \in M$ and $z \in Z$) are measurable.

We say that $\Phi$ is *integrable on $Z$* if for every $x \in M$, the function

$$z \mapsto |\Phi|_{(x,z)} = \sup\{|\Phi_{(x,z)}(v_1, \ldots, v_k)| : v_i \in T_xM \text{ for } i = 1, \ldots, k, \text{ with } \|v_i\|_x = 1\}$$

belongs to $L^1(Z,\mu)$. In this case we can consider the $k$-form

$$\omega_x(v_1,\ldots,v_k) = \left(\int_Z \Phi_{(x,z)} d\mu(z)\right)(v_1,\ldots,v_k) = \int_Z \Phi_{(x,z)}(v_1,\ldots,v_k) d\mu(z). \tag{2.2}$$

Observe that for all $x \in M$,

$$|\omega|_x \leq \int_Z |\Phi|_{(x,z)} d\mu(z) = \|\Phi_{(x,\cdot)}\|_{L^1}.$$

**Lemma 2.3.1.** *Let $\Phi : M \times Z \to \Lambda^k(M)$ be a measurable family of $k$-forms such that:*

- *It is integrable on $Z$, then we can define $\omega$ as in (2.2).*
- *For every fixed $z \in Z$ the $k$-form $x \mapsto \Phi_{(x,z)}$ is locally integrable and has weak derivative $d\Phi_{(x,z)}$.*
- *The function $z \mapsto |d\Phi|_{(x,z)}$ belongs to $L^1(Z,\mu)$ for every $x \in M$.*

*Then $\omega$ is locally integrable and has weak derivative*

$$d\omega_x = \int_Z d\Phi_{(x,z)} d\mu(z). \tag{2.3}$$

The previous lemma follows directly from definition of weak derivative.

To prove that a $k$-form $\omega : M \to \Lambda^k(M)$ is smooth it is enough to verify that for every set of $k$ vector fields $\{X_1,\ldots,X_k\}$ the function

$$f(x) = \omega_x(X_1(x),\ldots,X_k(x))$$

is smooth on $M$. A sufficient condition for $f$ to be smooth is that for every set of vector fields $\{Y_1,\ldots,Y_m\}$ there exists

$$L_{Y_m}\cdots L_{Y_1} f(x)$$

for all $x \in M$. The *Lie derivative* with respect to the field $Y$ is defined by

$$L_Y f(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} f(\varphi_t(x)),$$

where $\varphi_t$ is the flow associated to $Y$.

From the above observation and the classical Leibniz Integral Rule one can conclude the following lemma:

**Lemma 2.3.2.** *Let $M$ and $N$ be two Riemannian manifolds and $\Phi : M \times N \to \Lambda^k(M)$ a smooth family of $k$-forms on $M$, i.e. a measurable family of $k$-forms which coefficients are smooth functions on both variables. Suppose that one of the following conditions holds:*

*(i)* *For every $x \in M$, the form $y \mapsto \Phi_{(x,y)}$ has compact support.*

*(ii)* *There is an isometric embedding $\iota : N \to \tilde{N}$, such that $\iota(N)$ is an open subset of the Riemannian manifolds $\tilde{N}$ with compact closure, and $\Phi$ is a restriction of a smooth family of $k$-forms $\tilde{\Phi} : M \times \tilde{N} \to \Lambda^k(M)$.*

*Then the $k$-form $\omega$ defined as in (2.2) is smooth and*

$$d\omega_x = \int_N d\Phi_{(x,y)} dV_N(y),$$

*where $d\Phi_{(x,y)}$ denotes the derivative of the differential $k$-form $x \mapsto \Phi_{(x,y)}$ for a fixed $y \in N$.*

If $\omega$ is an integrable $n$-form in $L^1(M, \Lambda^n)$ and $M$ is orientable, one can define its integral on $M$ in the classical way, which satisfy

$$\left| \int_M \omega \right| \leq \|\omega\|_{L^1}.$$

As we have mentioned before we have the Hölder's inequality in the case of measurable forms:

**Lemma 2.3.3.** *Let $M$ be a Riemannian manifold, and $p$ and $q$ two real numbers in $(1, +\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then if we take $\omega \in L^p(M, \Lambda^k)$ and $\beta \in L^p(M, \Lambda^{n-k})$, with $k = 0, \dots, n$, the form $\omega \wedge \beta$ is integrable and*

$$\|\omega \wedge \beta\|_{L^1} \leq \|\omega\|_{L^p} \|\beta\|_{L^q}.$$

To prove the previous lemma is enough to observe that $|\omega \wedge \beta|_x \leq |\omega|_x |\beta|_x$ for all $x \in M$ and use the classic Hölder's inequality.

The *contraction* of a $k$-form $\omega$ on $M$ with respect to a vector field $Y$ is the $(k-1)$-form defined by

$$\iota_V \omega_x(v_1, \dots, v_{k-1}) = \omega_x(Y(x), v_1, \dots, v_{k-1})$$

for all $x \in M$ and $v_1, \dots, v_{k-1} \in T_x M$.

We have the following version of Fubini's theorem:

**Lemma 2.3.4.** *Let $M$ be an orientable smooth manifold of dimension $n$ and $I \subset \mathbb{R}$ an interval. Denote by $\frac{\partial}{\partial t}$ the field on $I \times M$ defined by*

$$\frac{\partial}{\partial t}(s, x) = (1, 0) \in \mathbb{R} \times T_x M = T_{(s,x)}(I \times M).$$

*If $\omega$ is an integrable $(n+1)$-form on $I \times M$, then*

$$\int_{I \times M} \omega = \int_I \left( \int_M \eta_s^* \left( \iota_{\frac{\partial}{\partial t}} \omega \right) \right) ds,$$

*where $\eta_s : M \to I \times M$, $\eta_s(x) = (s, x)$.*

## 2.4 Construction of non-zero classes using duality

Let us consider the following result mentioned in Section 1.1:

**Lemma 2.4.1** ([Pan08])**.** *Consider $M$ a complete orientable Riemannian manifold of dimension $n$ and two real numbers $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\omega$ be a $L^p$-integrable closed differential $k$-form on $M$. Then*

*(i)* *$\omega$ represents a non-zero class in $L^p\overline{H}^k(M)$ if, and only if, there exists a closed $(n-k)$-form $\beta \in L^q\Omega^{n-k}(M)$ such that $\int_M \omega \wedge \beta \neq 0$.*

*(ii)* *$\omega$ represents a non-zero class in $L^pH^k(M)$ if, and only if, there exists a sequence of differential $(n-k)$-forms $\{\beta_j\}_{j\in\mathbb{N}}$ such that*

$$\int_M \omega \wedge \beta_j \geq 1 \text{ and } \|d\beta_j\|_{L^q} \to 0.$$

We use the previous lemma to construct non-zero cohomology classes for some examples.

**Example 2.4.2.** Consider the real hyperbolic space $\mathbb{H}^n = \mathbb{R}^{n-1} \times (0, +\infty)$ with the metric given by

$$\langle v, w\rangle_{(x,t)} = \frac{v_1w_1 + \cdots + v_nw_n}{t^2},$$

for $(x,t) \in \mathbb{H}^n$ and $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ two vectors in $T_{(x,t)}\mathbb{H}^n = \mathbb{R}^n$. Take the closed differential forms

$$\omega_{(x,t)} = d(f(x)g(t)dx_1 \wedge \cdots \wedge dx_{k-1}) \text{ and } \beta_{(x,t)} = d(\psi(x)\varphi(t)\pi_k(x)dx_{k+1} \wedge \cdots \wedge dx_{n-1}),$$

where $\pi_k : \mathbb{R}^{n-1} \to \mathbb{R}$ is the projection onto the $k^{th}$ coordinate, and $f, \psi : \mathbb{R}^{n-1} \to [0,1]$ and $g, \varphi : (0, +\infty) \to [0,1]$ are smooth functions such that

- $f$ has compact support and $\int_{\mathbb{R}^{n-1}} f(x)dx = 1$.
- $g(t) = 0$ if $t$ is bigger than some $t_1 > 0$ and $g(t) = 1$ if $t$ is smaller than some $t_0 > 0$.
- $\varphi(t) = 0$ if $t$ is bigger than $t_2 > 0$ and $\varphi(t) = 1$ if $t$ is smaller than $t_1$.
- $\psi$ is constant 1 on the support of $f$ and has compact support.

Observe that

$$\|\omega\|_{L^p} \leq \|fg'\ dt \wedge dx_1 \wedge \cdots \wedge dx_{k-1}\|_{L^p} + \sum_{i=k}^{n-1} \left\| \frac{\partial f}{\partial x_i} g\ dx_i \wedge dx_1 \wedge \cdots \wedge dx_{k-1} \right\|_{L^p}.$$

The first term is finite because $fg'$ has compact support. Using that $|dx_{i_1}\wedge\cdots\wedge dx_{i_k}| = t^k$ we can estimate the others:

$$\begin{aligned}\left\|\frac{\partial f}{\partial x_i} g\ dx_i \wedge dx_1 \wedge \cdots \wedge dx_{k-1}\right\|_{L^p}^p &= \int_{\mathbb{H}^n} \left|\frac{\partial f}{\partial x_i}(x)\right|^p g(t)^p t^{pk}\ dV(x,t) \\ &= \int_0^{+\infty}\int_{\mathbb{R}^{n-1}} \left|\frac{\partial f}{\partial x_i}(x)\right|^p g(t)^p t^{pk-n} dx dt \\ &\leq \left\|\frac{\partial f}{\partial x_i}\right\|_{L^p}^p \int_0^{t_1} t^{pk-n} dt.\end{aligned}$$

This is finite if $p > \frac{n-1}{k}$. Moreover

$$\begin{aligned}|\beta|_{(x,t)} \leq& \sum_{i=1}^k \left|\frac{\partial \psi}{\partial x_i}(x)\varphi(t)\pi_k(x)\right|\ |dx_i \wedge dx_{k+1}\wedge\cdots\wedge dx_{n-1}|_{(x,t)} \\ &+ |\psi(x)\varphi(t)|\ |dx_k \wedge \cdots \wedge dx_{n-1}|_{(x,t)} \\ &+ |\psi(x)\pi_k(x)\varphi'(t)|\ |dt \wedge dx_{k+1}\wedge\cdots\wedge dx_{n-1}|_{(x,t)} \\ &= t^{n-k}\left(\sum_{i=1}^k \left|\frac{\partial \psi}{\partial x_i}(x)\varphi(t)\pi_k(x)\right| + |\psi(x)\varphi(t)| + |\psi(x)\pi_k(x)\varphi'(t)|\right).\end{aligned}$$

From this one can see that $\|\beta\|_{L^q}$ is finite for every $q > \frac{n-1}{n-k}$, which is equivalent to $p < \frac{n-1}{k-1}$.

If $p \in \left(\frac{n-1}{k}, \frac{n-1}{k-1}\right)$ the $n$-form $\omega \wedge \beta$ is integrable by Holder's inequality; hence by Stokes theorem we have

$$\begin{aligned}\int_{\mathbb{H}^n} \omega\wedge\beta &= \lim_{s\to 0}\int_{\mathbb{R}^{n-1}\times(s,t_1)} \omega\wedge\beta \\ &\leq \lim_{s\to 0}\int_{\mathbb{R}^{n-1}\times\{s\}} f\ dx_1\wedge\cdots\wedge dx_{n-1} = 1.\end{aligned}$$

By Lemma 2.4.1 we have that the reduced $L^p$-cohomology in degree $k \geq 2$ is not zero for all $p \in \left(\frac{n-1}{k}, \frac{n-1}{k-1}\right)$. This interval is a maximal open interval in $\mathrm{Ann}_k^c(\mathbb{H}^n) = [1,+\infty)\setminus \mathrm{Ann}_k(\mathbb{H}^n)$. In fact, if $p \notin \left[\frac{n-1}{k}, \frac{n-1}{k-1}\right]$, then $L^pH^k(\mathbb{H}^n) = 0$ (see for example [Bou16, Corollary B]).

In the case $k = 1$ we can look at [BP03, Theorem 0.3]. Since there exists an Ahlfors-regular visual metric of dimension $n-1$ on $\partial\mathbb{H}^n$ such that it is a Loewner space, then $L^pH^1(\mathbb{H}^n) = 0$ if, and only if, $p \leq n-1$.

Observe that the real hyperbolic space $\mathbb{H}^n$ is isometric to the group $\mathbb{R}^{n-1}\rtimes_{Id}\mathbb{R}$ with the left-invariant metric induced by the Euclidean inner product on the tangent space at the identity.

**Example 2.4.3.** Consider now the Heintze group $G = \mathbb{R}^3 \rtimes_\alpha \mathbb{R}$ with

$$\alpha = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 3 \end{pmatrix}.$$

We equip $G$ with the left-invariant metric induced by the Euclidean inner product on the tangent space at the unity, which is given by

$$\langle v, w\rangle_{(x,t)} = e^{-t}v_1w_1 + e^{-t}v_2w_2 + e^{-3t}v_3w_3 + v_4w_4.$$

The volume form on $G$ is $dV(x,t) = e^{-5t}dxdt$, and the operator norms of the fundamental horizontal forms on $G$ are:

- $|dx_1|_{(x,t)} = |dx_2|_{(x,t)} \asymp e^t$, $|dx_3|_{(x,t)} \asymp e^{3t}$,
- $|dx_1 \wedge dx_2|_{(x,t)} \asymp e^{2t}$, $|dx_1 \wedge dx_3|_{(x,t)} = |dx_2 \wedge dx_3|_{(x,t)} \asymp e^{4t}$, and
- $|dx_1 \wedge dx_2 \wedge dx_3| \asymp e^{5t}$

See Lemma 3.4.4 for more details about these estimates.

We can consider the forms

$$\omega_{(x,t)} = d(f(x)g(t)dx_1 \wedge dx_2) \text{ and } \beta_{(x,t)} = d(\psi(x)\varphi(t)\pi_3(x)),$$

as in the Example 2.4.2 (the functions $g, \varphi$ are extended to $\mathbb{R}$ by putting $g(t) = \varphi(t) = 1$ for all $t < 0$). Using the same argument as above we can prove that $L^p\overline{H}^3(G) \neq 0$ for all $p \in \left(1, \frac{5}{4}\right)$. But this interval is not maximal in $\mathrm{Ann}_3^c(G)$. Indeed, because of [Pan08, Proposition 27] the interval $\left[\frac{5}{4}, \frac{5}{2}\right)$ is contained in $\mathrm{Ann}_3^c(G)$ and $\left(\frac{5}{2}, +\infty\right) \subset \mathrm{Ann}_3(G)$.

In order to improve the result we can try exchanging $\beta$ by a sequence $\{\beta_j\}_{j\in\mathbb{N}}$ satisfying the second condition of Lemma 2.4.1.

Consider $\beta_j = \Psi_j\ dx_3$, where $\Psi_j(x,t) = 1$ if $(x,t) \in \mathrm{supp}(fg)$. We suppose that $\beta_j \in L^q\Omega^1(G)$ for every $j \in \mathbb{N}$ and $\|d\beta_j\|_{L^q} \to 0$.

Observe that

$$\|d\beta_j\|_{L^q} \geq \left\| \frac{\partial \Psi_j}{\partial x_i}\ dx_i \wedge dx_3 \right\|_{L^q} \quad (i = 1, 2). \tag{2.4}$$

Moreover

$$\begin{aligned} \left\| \frac{\partial \Psi_j}{\partial x_i}\ dx_i \wedge dx_3 \right\|_{L^q}^q &= \int_G \left| \frac{\partial \Psi_j}{\partial x_i}(x,t) \right|^q |dx_i \wedge dx_3|_{(x,t)}^q dV(x,t) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \left\| \frac{\partial \Psi_j}{\partial x_i}(\cdot,\cdot,x_3,t) \right\|_{L^q(\mathbb{R}^2)}^q e^{t(4q-5)} dx_3 dt. \end{aligned}$$

Since $\beta_j \in L^q\Omega^1(G)$, the function $h^j_{(x_3,t)} = \Psi_j(\cdot,\cdot,x_3,t)$ belongs to the Sobolev space $W^{1,q}(\mathbb{R}^2)$ for almost every $x_3$ and $t$. Then if $q < 2$ the first Sobolev inequality says that there exists a constant $C > 0$ depending on $q$ such that

$$\|h^j_{(x_3,t)}\|_{L^{q^*}(\mathbb{R}^2)} \leq C\|\nabla h^j_{(x_3,t)}\|_{L^q(\mathbb{R}^2)}$$

for almost every $x_3$ and $t$, where $q^* = 2q/2 - q$ (see for example [Hei01]). Thus

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \|h^j_{(x_3,t)}\|_{L^{q^*}(\mathbb{R}^2)} e^{t(4q-5)} dx_3 dt \leq C \int_{\mathbb{R}}\int_{\mathbb{R}} \|\nabla h^j_{(x_3,t)}\|_{L^q(\mathbb{R}^2)} e^{t(4q-5)} dx_3 dt \to 0,$$

which contradicts the assumption that $\Psi_j \equiv 1$ on the support of $fg$. For this reason we find this method useless to construct non-zero classes for $p \in \left(2, \frac{5}{2}\right)$.

## 2.5 Orlicz spaces and doubling Young functions

In this section we see some properties of Orlicz spaces that will be useful in Chapter 4. Consider $\phi$ a Young function and $(Z, \mu)$ a measure space.

**Remark 2.5.1.** If $K \geq 1$ is any constant, the identity map $\mathrm{Id} : L^{K\phi}(Z,\mu) \to L^{\phi}(Z,\mu)$ is clearly continuous and bijective, thus it is an isomorphism by the open mapping theorem. This implies that the norms $\| \ \|_{L^{K\phi}}$ and $\| \ \|_{L^{\phi}}$ are equivalent for all $K > 0$.

**Lemma 2.5.2.** *If $\mu$ is finite, then $L^{\phi}(Z,\mu) \subset L^1(Z,\mu)$ and the inclusion is continuous, with norm bounded depending only on $\mu(Z)$ and $\phi$.*

*Proof.* Let $f \in L^{\phi}(Z,\mu)$, then

$$\begin{aligned}
\|f\|_{L^\phi} &= \inf\left\{\gamma > 0 : \int_Z \phi\left(\frac{f}{\gamma}\right) d\mu \leq 1\right\} \\
&\geq \inf\left\{\gamma > 0 : \mu(Z)\phi\left(\frac{1}{\mu(Z)}\int_Z \frac{f}{\gamma} d\mu\right) \leq 1\right\}
\end{aligned}$$

From this we obtain $\|f\|_{L^1} \leq \mu(Z)\phi^{-1}(1/\mu(Z))\|f\|_{L^\phi}$. □

Remember the definition of doubling Young function that we give in Section 1.2.3. It is not difficult to prove the following equivalence.

**Lemma 2.5.3.** *A Young function $\phi$ is doubling if, and only if, there exists an increasing function $D_1 : [2,+\infty) \to (1,+\infty)$ such that for all $t \in \mathbb{R}$ and $s \in [2,+\infty)$,*

$$\phi(st) \leq D_1(s)\phi(t).$$

There are some special properties that have Orlicz spaces associated to doubling Young functions, as we can see in the following lemma:

**Lemma 2.5.4.** *Let $\phi$ a doubling Young function, then*

*(i) $f \in L^\phi(Z,\mu)$ if, and only if, $\int_Z \phi(f)d\mu < +\infty$.*

*(ii) $f_n \to f$ in $L^\phi(Z,\mu)$ if, and only if, $\int_Z \phi(f_n - f)d\mu \to 0$.*

*Proof.* (i) ($\Rightarrow$) Since $f \in L^\phi(Z,\mu)$ there exists $\gamma \geq 2$ such that $\int_Z \phi\left(\frac{f}{\gamma}\right) d\mu \leq +\infty$. Then

$$\int_Z \phi(f)d\mu \leq D_1(1/\gamma) \int_Z \phi\left(\frac{f}{\gamma}\right) d\mu \leq +\infty,$$

where $D_1$ is the function given in Lemma 2.5.3.

($\Leftarrow$) We have that $\phi\left(\frac{f}{\gamma}\right) \leq \phi(f) \in L^1(Z,\mu)$ for all $\gamma \in [1,+\infty)$, and that $\phi\left(\frac{f}{\gamma}\right) \to 0$ almost everywhere when $\gamma \to +\infty$. Because of the Dominated Convergence Theorem we have

$$\int_Z \phi\left(\frac{f}{\gamma}\right) d\mu \to 0, \text{ when } \gamma \to +\infty.$$

Thus $f \in L^\phi(Z,\mu)$.

(ii) ($\Rightarrow$) Suppose that $\|f_n - f\|_{L^\phi} < 1$. Using the convexity of $\phi$ we obtain

$$\int_Z \phi(f_n - f)d\mu \leq \|f_n - f\|_{L^\phi} \int_Z \phi\left(\frac{f_n - f}{\|f_n - f\|_{L^\phi}}\right) d\mu \leq \|f_n - f\|_{L^\phi} \to 0.$$

($\Leftarrow$) Given $\epsilon \in \left(0, \frac{1}{2}\right)$ we have

$$\int_Z \phi\left(\frac{f_n - f}{\epsilon}\right) d\mu \leq D_1(1/\epsilon) \int_Z \phi(f_n - f)d\mu \to 0.$$

This implies that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\int_Z \phi\left(\frac{f_n - f}{\epsilon}\right) d\mu \leq 1,$$

which means that $\|f_n - f\|_{L^\phi} < \epsilon$ for all $n \geq n_0$.

∎

# Chapter 3

# Relative $L^p$-Cohomology and Application

We begin this chapter working with the simplicial relative $\ell^p$-cohomology. In the first section we prove Theorem 1.2.1. Then, in the second section, we explain how to construct a simplicial pair associated to a Gromov-hyperbolic Riemannian manifold with bounded geometry and a point on its boundary. We prove Theorem 1.2.3 after some previous lemmas.

If $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{p} = 1$, there exists a duality relationship between $L^p$ and $L^q$-cohomology in the classical sense. There are some difficulties to adapt this result to our relative version, however we can give some ideas related to this subject. This is the contents of Section 3.3.

Finally we study the $L^p$-cohomology of a purely real Heintze group of the form $G = \mathbb{R}^{n-1} \rtimes_\alpha \mathbb{R}$ relative to the point $\infty \in \partial G$, which allows us to prove Theorem 1.2.7.

## 3.1 Quasi-isometry invariance of simplicial relative $\ell^p$-cohomology

Consider $X$ a finite-dimensional simplicial complex with bounded geometry. Observe that every element $\theta \in \ell^p C^k(X)$ has a natural linear extension $\theta : C_k(X) \to \mathbb{R}$, where

$$C_k(X) = \left\{ \sum_{i=1}^{m} t_i \sigma_i : t_1, \dots, t_m \in \mathbb{R}, \sigma_1, \dots, \sigma_m \in X^k \right\}.$$

The *support* of a chain $c = \sum_{i=1}^{m} t_i \sigma_i$ in $C_k(X)$, with $t_i \neq 0$ for all $i = 1, \dots, m$, is $|c| = \{\sigma_1, \dots, \sigma_m\}$. We also define the *uniform norm* and the *length* of $c$ by

$$\|c\|_\infty = \max\{|t_1|, \dots, |t_m|\}, \text{ and } \ell(c) = m.$$

**Proposition 3.1.1.** *The coboundary operator $\delta : \ell^p C^k(X) \to \ell^p C^{k+1}(X)$ is well-defined and continuous.*

*Proof.* Let $\theta$ be a $k$-cochain in $\ell^p C^k(X)$, then

$$\begin{aligned}\|\delta_k(\theta)\|_{\ell^p}^p &= \sum_{\sigma\in X^{k+1}} |\delta_k(\theta)(\sigma)|^p = \sum_{\sigma\in X^{k+1}} |\theta(\partial\sigma)|^p = \sum_{\sigma\in X^{k+1}} \left|\theta\left(\sum_{\tau\in|\partial\sigma|}\tau\right)\right|^p \\ &= (k+2)^p \sum_{\sigma\in X^{k+1}} \left|\left(\sum_{\tau\in|\partial\sigma|} \frac{\theta(\tau)}{k+2}\right)\right|^p \leq (k+2)^{p-1} \sum_{\sigma\in X^{k+1}} \sum_{\tau\in|\partial\sigma|} |\theta(\tau)|^p \\ &\leq N(1)(k+2)^{p-1} \sum_{\tau\in X^k} |\theta(\tau)|^p = \|\theta\|_{\ell^p}^p,\end{aligned}$$

where the first inequality follows form Jensen's inequality and the second is a consequence of bounded geometry. The function $N$ is as in the definition of bounded geometry, which implies that $N(1)$ is a bound for the number of simplices $\sigma \in X^{k+1}$ such that $|\partial\sigma|$ contains a fixed simplex $\tau \in X^k$. $\square$

As we said, we will prove Theorem 1.2.1 in a similar way as in [BP03]. For this purpose we first prove some lemmas.

**Lemma 3.1.2** ([BP03])**.** *Let $X$ and $Y$ be two uniformly contractible simplicial complexes with bounded geometry. Then, any quasi-isometry $F : X \to Y$ induces a family of maps $c_F : C_k(X) \to C_k(Y)$ verifying:*

*(i) $\partial c_F(\sigma) = c_F(\partial\sigma)$ for every $\sigma \in X^k$.*

*(ii) For every $k \in \mathbb{N}$ there exist two constants $N_k$ and $L_k$ (depending on $k$ and the geometric data of $X, Y$ and $F$) such that*

$$\|c_F(\sigma)\|_\infty \leq N_k, \text{ and } \ell(c_F(\sigma)) \leq L_k,$$

*for all $\sigma \in X^k$.*

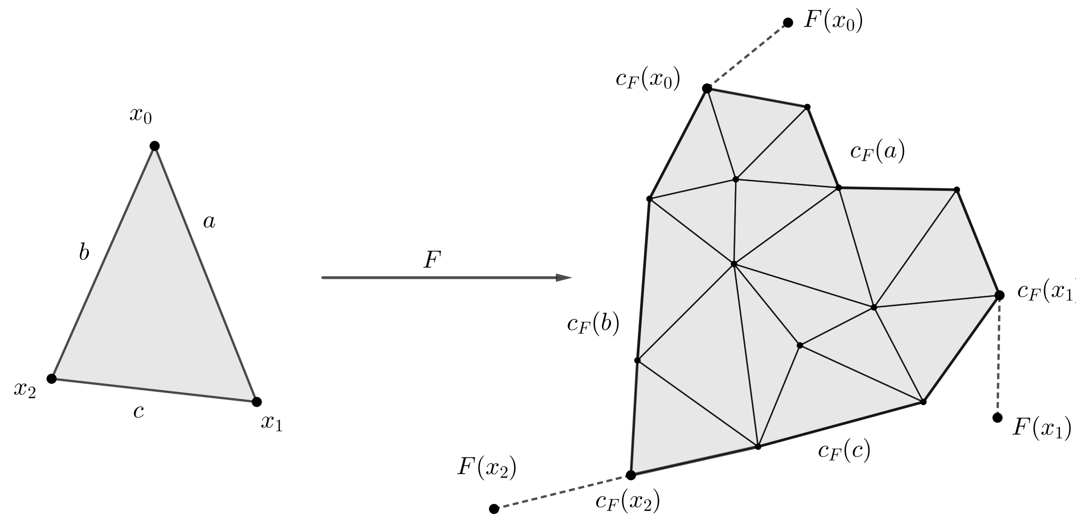
*Proof.* We consider for both complexes $X$ and $Y$ the same constant $C \geq 0$ and function $N : [0, +\infty) \to \mathbb{N}$ corresponding to their bounded geometry. We assume also that both spaces are uniformly contractible for the same function $\psi$.

For $v \in X^0$ we define $c_F(v)$ as a vertex of a simplex containing $F(v)$. Because of the bounded geometry we have $|F(v) - c_F(v)| \leq C$. We extend linearly $c_F$ to $C_0(X)$.

Since $F$ is a quasi-isometry and $X$ has bounded geometry, then

$$\sup\{|c_F(a_+) - c_F(a_-)| : a \in X^1\} < +\infty,$$

where $a_-$ and $a_+$ denote the vertices of $a$. This supremum depends only on the geometric data of $X$, $Y$ and $F$. Using again the bounded geometry of $Y$ we can find a chain $c_F(a) \in C_1(Y)$ with $\partial c_F(a) = c_F(a_+) - c_F(a_-)$, length bounded by a function of $|c_F(a_+) - c_F(a_-)|$, and $\|c_F(a)\|_\infty = 1$.




Now we take $k \geq 2$. Assume that $c_F$ is defined in degree $m$ for every $m \leq k-1$. If $\sigma \in X^k$, then $c_F(\partial\sigma)$ is a cycle ($\partial c_F(\partial\sigma) = c_F(\partial^2\sigma) = 0$) and it is contained in a ball with radius $kCL_{k-1}$. Since $Y$ is uniformly contractible, $c_F(\partial\sigma)$ is the boundary of a chain contained in a ball $B$ with radius $\psi(kCL_{k-1})$. Its length is bounded by the number of simplices in $B$, which is less than $N(\psi(kCL_{k-1}))$. We define $c_F(\sigma)$ as such a chain that minimize $\|c_F(\sigma)\|_\infty$. Since $\|c_F(\partial\sigma)\|_\infty \leq kN_{k-1}$ we have that $\|c_F(\sigma)\|_\infty$ is bounded independently on $\sigma$. □

**Lemma 3.1.3** ([BP03]). *Consider $F, G : X \to Y$ two quasi-isometries between uniformly contractible simplicial complexes with bounded geometry. If $F$ and $G$ are at bounded uniform distance, then there exists an homotopy $h : C_k(X) \to C_{k+1}(Y)$ between $c_F$ and $c_G$. This means that*

*(i) $\partial h(v) = c_F(v) - c_G(v)$ if $v \in X^0$, and*

*(ii) $\partial h(\sigma) + h(\partial\sigma) = c_F(\sigma) - c_G(\sigma)$ if $\sigma \in X^k$, $k \geq 1$.*

*Moreover, $\|h(\sigma)\|_\infty$ and $\ell(h(\sigma))$ are uniformly bounded by constants $N'_k$ and $L'_k$ that only depend on the geometric data of $X, Y, F$ and $G$.*

*Proof.* Since $F$ and $G$ are at bounded uniform distance, for all $k \geq 0$,

$$\sup\{\operatorname{diam}(|c_F(\sigma)| \cup |c_G(\sigma)|) : \sigma \in X^k\} < +\infty.$$

If $v$ is a vertex in $X^0$ we choose a chain $h(v)$ such that $\partial h(v) = c_F(v) - c_G(v)$ with length bounded depending on $|c_F(v) - c_G(v)|$ and $\|h(v)\|_\infty = 1$. Note that it is possible using an argument as in the previous lemma.

Suppose that $h$ is defined in degree $m$ for every $m \leq k-1$ and consider $\sigma \in X^k$. Since $c_F$ and $c_G$ commute with the boundary, we have

$$\partial(c_G(\sigma) - c_F(\sigma) - h(\partial\sigma)) = c_G(\partial\sigma) - c_F(\partial\sigma) - \partial h(\partial\sigma) = 0.$$

This means that $c_G(\sigma) - c_F(\sigma) - h(\partial\sigma)$ is a cycle contained in a ball with radius bounded independently of $\sigma \in X^k$. As in the previous lemma we can find $h(\sigma) \in C_{k+1}(Y)$ with boundary $c_G(\sigma) - c_F(\sigma) - h(\partial\sigma)$, and $\ell(h(\sigma))$ and $\|h(\sigma)\|_\infty$ uniformly bounded. $\square$

Now assume that $X$ is Gromov-hyperbolic. We are ready to prove the invariance of relative $\ell^p$-cohomology.

*Proof of Theorem 1.2.1.* We define the *pull-back* of a cochain $\theta \in \ell^p_{F(\xi)} C^k(Y)$ as

$$F^*\theta = \theta \circ c_F.$$

Observe that $F^*$ depends on the choice of $c_F$.

Let us first show that $F^*\theta \in \ell^p C^k(X)$:

$$\begin{aligned}
\|F^*\theta\|^p_{L^p} &= \sum_{\sigma \in X^k} |F^*\theta(\sigma)|^p = \sum_{\sigma \in X^k} |\theta(c_F(\sigma))|^p \\
&\leq \sum_{\sigma \in X^k} N_k^p \left| \sum_{\tau \in |c_F(\sigma)|} \theta(\tau) \right|^p \\
&\leq N_k^p \sum_{\sigma \in X^k} \sum_{\tau \in |c_F(\sigma)|} \ell(c_F(\sigma))^{p-1} |\theta(\tau)|^p \\
&\leq N_k^p L_k^{p-1} \sum_{\sigma \in X^k} \sum_{\tau \in |c_F(\sigma)|} |\theta(\tau)|^p.
\end{aligned}$$

Since $F$ is a quasi-isometry and the distance between $c_F(v)$ and $F(v)$ is uniformly bounded for all $v \in X^0$, we can find a constant $C_k$ such that if $dist(\sigma_1, \sigma_2) > C_k$, then $c_F(\sigma_1) \cap c_F(\sigma_2) = \emptyset$. Using the bounded geometry of $X$ we have that every $\tau \in Y^k$ satisfies $\tau \in |c_F(\sigma)|$ for at most $N(C + C_k)$ simplices $\sigma \in X^k$. This implies that

$$\begin{aligned}
\|F^*\theta\|^p_{L^p} &\leq N_k^p L_k^{p-1} N(C + C_k) \sum_{\tau \in Y^k} |\theta(\tau)|^p \\
&= N_k^p L_k^{p-1} N(C + C_k) \|\theta\|^p_{L^p},
\end{aligned}$$

This also proves the continuity of $F^*$.

Now we prove that for every $\theta$ in $\ell^p_{F(\xi)}C^k(Y)$, the cochain $F^*\theta$ is zero on some neighborhood of $\xi$. Assume that $\theta$ is zero on $V \subset \overline{Y}$, a neighborhood of $F(\xi)$. If $\sigma \in X^k$ and $v \in X^0$ is a vertex of $\sigma$,

$$d_H(c_F(\sigma), F(v)) \leq d_H(c_F(\sigma), c_F(v)) + d_H(c_F(v), F(v)), \tag{3.1}$$

where $d_H$ denotes the Hausdorff distance. By construction of $c_F$, distance (3.1) is uniformly bounded by a constant $\tilde{C}_k$. We define $\tilde{V} = \{y \in Y : dist(y, V^c \cap Y) > \tilde{C}_k\}$. Since $F$ is a quasi-isometry, there exists $U \subset \overline{X}$ a neighborhood of $\xi$ such that $F(U \cap X) \subset \tilde{V}$. For every $k$-simplex $\sigma \subset U$, we have $c_F(\sigma) \subset V$ and then $F^*\theta(\sigma) = 0$. We conclude that $F^*\theta$ vanishes on $U$.

By definition we have $\delta F^* = F^*\delta$, which implies that $F^*$ defines a map in cohomology denoted by $F^\# : \ell^p_{F(\xi)}H^k(Y) \to \ell^p_\xi H^k(X)$. We have to prove that $F^\#$ is an isomorphism.

Claim: If $F, G : X \to Y$ are two quasi-isometries at bounded uniform distance, then $F^\# = G^\#$.

We have to construct a family of continuous linear maps $H_k : \ell^p_{F(\xi)}C^k(Y) \to \ell^p_\xi C^{k-1}(X)$, $k \geq 1$, such that

(i) $F^*\theta - G^*\theta = H_1\delta\theta$ for all $\theta \in \ell^p_{F(\xi)}C^0(Y)$.

(ii) $F^*\theta - G^*\theta = H_{k+1}\delta\theta + \delta H_k\theta$ for all $\theta \in \ell^p_{F(\xi)}C^k(Y)$, $k \geq 1$.

We define

$$H_k\theta : X^k \to \mathbb{R},\ H_k\theta(\sigma) = \theta(h(\sigma)),$$

where $h$ is the map defined in Lemma 3.1.3. Using the same argument as for $F^*$, we can prove that $H_k\theta$ is in $\ell^pC^{k-1}(X)$ for every $\theta \in \ell^p_{F(\xi)}C^k(Y)$ and $H_k$ is continuous. To see that $H_k\theta$ vanishes on some neighborhood of $\xi$, observe that $h(\sigma)$ have uniformly bounded length, which implies that $d_H(c_F(\sigma), h(\sigma))$ is uniformly bounded.

Moreover, if $k = 0$ we have

$$(F^*\theta - G^*\theta)(v) = \theta(c_F(v) - c_G(v)) = \theta(\partial h(v)) = \delta\theta(h(v)) = H_1\delta\theta(v).$$

And if $k \geq 1$,

$$\begin{aligned}(F^*\theta - G^*\theta)(\sigma) &= \theta(c_F(\sigma) - c_G(\sigma)) \\ &= \theta(\partial h(\sigma) + h(\partial\sigma)) \\ &= \delta\theta(h(\sigma)) + \theta(h(\partial\sigma)) \\ &= H_{k+1}\delta\theta(\sigma) + H_k\theta(\partial\sigma) \\ &= H_{k+1}\delta\theta(\sigma) + \delta H_k\theta(\sigma).\end{aligned}$$

This proves the claim.

As a consequence of the claim we have that $F^{\#}$ does not depend on the choice of $c_F$. Moreover, if $T : Y \to Z$ is another quasi-isometry, a possible choice of the function $c_{T\circ F}$ is the composition $c_T \circ c_F$. In this case $(T \circ F)^* = F^* \circ T^*$ and then $(T \circ F)^{\#} = F^{\#} \circ T^{\#}$.

Finally, if $\overline{F} : Y \to X$ is a quasi-inverse of $F$, then by the claim $(F \circ \overline{F})^{\#}$ and $(\overline{F} \circ F)^{\#}$ are the identity in relative cohomology. Since $(F \circ \overline{F})^{\#} = \overline{F}^{\#} \circ F^{\#}$ and $(\overline{F} \circ F)^{\#} = F^{\#} \circ \overline{F}^{\#}$, the statement follows. □

Let us see a simple example of $\ell^p$-cohomology of a Gromov-hyperbolic simplicial complex relative to two diferent boundary points.

**Example 3.1.4.** We consider for $n \geq 3$ the space

$$X = \frac{\mathbb{H}^n \cup [0, +\infty)}{x_0 \sim 0},$$

where $x_0$ is a point in $\mathbb{H}^n$. We equip $X$ with the length distance making the inclusions $\mathbb{H}^n \hookrightarrow X$ and $[0, +\infty) \hookrightarrow X$ isometric embeddings. A triangulation of $\mathbb{H}^n$ with bounded geometry (for which $x_0$ is a vertex) and the usual graph structure on $[0, +\infty)$ (where $\mathbb{N}$ is the set of vertices) induce a simplicial structure on $X$ with bounded geometry and uniformly contractible. Observe that $X$ is a Gromov-hyperbolic space and its boundary can be write $\partial X = \mathbb{S}^{n-1} \cup \{\xi_0\}$, where the sphere $\mathbb{S}^{n-1}$ is identified with the boundary of $\mathbb{H}^n$ and $\xi_0$ is the point corresponding to the geodesic ray $[0, +\infty)$.

As we saw at the end of Example 2.4.2, $\ell^p H^1(\mathbb{H}^n) = 0$ if $p \in (1, n-1)$. From this it is easy to see that $\ell^p H^1(\mathbb{H}^n, \xi) = 0$ for every $\xi \in \partial\mathbb{H}^n$ and $p \in (1, n-1)$. We will use these facts to prove that $\ell^p H^1(X, \xi)$ is not isomorphic to $\ell^p H^1(X, \xi_0)$ for every $p \in (1, n-1)$ an $\xi \in \partial\mathbb{H}^n$.

Consider in $\ell^p(\mathbb{N})$ the subspaces

- $\mathcal{V} = \{\{a_n\}_{n\in\mathbb{N}} : \exists\ n_0 \in \mathbb{N},\ a_n = 0\ \forall n \geq n_0\}$, and
- $W = \big\{\{a_n\}_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}) : \sum_{n\in\mathbb{N}} a_n = 0\big\}$.

We define the linear map $f : \mathcal{V} \to \ell^p C^1(X, \xi_0)$ by

$$f(\{a_n\}_{n\in\mathbb{N}})(e) = \begin{cases} 0 & \text{if } e \subset \mathbb{H}^n \\ a_n & \text{if } e = [n, n+1] \subset [0, +\infty) \end{cases}.$$

<u>Claim 1</u>: $f$ induces a linear isomorphism between $\mathcal{V}/(\mathcal{V} \cap W)$ and $\ell^p H^1(X, \xi_0)$.

First we prove that $f$ passes to the quotient. It is clear that $f(\{a_n\}_{n\in\mathbb{N}})$ is always a cocycle. Take $\{a_n\}_{n\in\mathbb{N}}$ in $\mathcal{V} \cap W$ and denote $\theta = f(\{a_n\}_{n\in\mathbb{N}})$. We define a 0-cochain

$\vartheta$ on $X$ by

$$\vartheta(x) = \begin{cases} 0 & \text{if } x \in \mathbb{H}^n \\ \sum_{i=0}^{n-1} a_i & \text{if } x = n \in [0,+\infty) \end{cases} .$$

By definition $\delta\vartheta = \theta$. Since $\{a_n\}_{n\in\mathbb{N}}$ belongs to $\mathcal{V} \cap W$, we have that $\vartheta(n) = 0$ if $n$ is big enough, then $\vartheta$ is in $\ell^p C^0(X, \xi_0)$. Therefore $\theta$ is zero in cohomology and $f$ induces a linear function

$$\tilde{f} : \mathcal{V}/(\mathcal{V} \cap W) \to \ell^p H^1(X, \xi_0).$$

It is easy to see that $\tilde{f}$ is inyective: Suppose that there exists $\vartheta \in \ell^p C^0(X, \xi_0)$ such that $\delta\vartheta = f(\{a_n\}_{n\in\mathbb{N}})$. This cochain must satisfy $\vartheta(0) = 0$ and $\vartheta(n) = 0$ if $n$ is big enough, which implies that $\{a_n\}_{n\in\mathbb{N}} \in W$.

Now we prove that $\tilde{f}$ is surjective. Take a 1-cocycle $\theta \in \ell^p C^1(X, \xi_0)$. Since $\ell^p H^1(\mathbb{H}^n) = 0$ there exists $\beta \in \ell^p C^0(\mathbb{H}^n)$ such that $\delta\beta = \theta|_{\mathbb{H}^n}$. Consider the following 1-cocycle in $\ell^p C^1(X, \xi_0)$:

$$\tilde{\theta}(e) = \begin{cases} \theta(e) & \text{if } e \subset \mathbb{H}^n \\ -\beta(0) & \text{if } e = [0,1] \\ 0 & \text{if } e = [n, n+1] \text{ with } n \geq 0 \end{cases} .$$

The cocycle $\tilde{\theta}$ is zero in $\ell^p H^1(X, \xi_0)$ and $\theta - \tilde{\theta}$ belongs to $\operatorname{Im} f$, thus $\pi(\theta) \in \operatorname{Im} \tilde{f}$ (where $\pi : \ell^p C^1(X, \xi_0) \to \ell^p H^1(X, \xi_0)$ is the canonical projection). This finishes the proof of the Claim 1.

A consequence of Claim 1 is that $\dim(\ell^p H^1(X, \xi_0)) = 1$ because $\mathcal{V} \cap W$ is the kernel of the linear map $\varphi : \mathcal{V} \to \mathbb{R}$ defined by

$$\varphi(\{a_n\}_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} a_n.$$

Now we want to study $\ell^p H^1(X, \xi)$ for $\xi \neq \xi_0$.

Claim 2: If $\theta \in \ell^p C^1(X, \xi)$ there exists a 1-cochain $\overline{\theta}$ such that $\pi(\theta) = \pi(\overline{\theta})$ and $\overline{\theta}|_{\mathbb{H}^n} \equiv 0$.

In the same way as we proved that $\tilde{f}$ is surjective we can find a cochain $\tilde{\theta} \in \ell^p C^1(X, \xi)$ such that $\pi(\tilde{\theta}) = 0$ and $\tilde{\theta}|_{\mathbb{H}^n} = \theta|_{\mathbb{H}^n}$ (here we use that $\ell^p H^1(\mathbb{H}^n, \xi) = 0$). Then we can take $\overline{\theta} = \theta - \tilde{\theta}$.

Let us consider the map $g : \ell^p H^1(X, \xi) \to \ell^p(\mathbb{N})/W$, where $g(\theta)$ is the class of $\{\overline{\theta}([n, n+1])\}_{n\in\mathbb{N}}$, denoted by $[\overline{\theta}([n, n+1])]_{n\in\mathbb{N}}$. Here $\overline{\theta}$ is as in Claim 2.

Claim 3: $g$ is well-defined, linear and surjective.

To prove that $g$ is well-defined it is enough to show that if $\overline{\theta}_1, \overline{\theta}_2 \in \ell^p C^1(X,\xi)$ are in the same cohomology class and $\overline{\theta}_1|_{\mathbb{H}^n} = \overline{\theta}_2|_{\mathbb{H}^n} \equiv 0$, then $\{\overline{\theta}_1 - \overline{\theta}_2([n,n+1])\}_{n\in\mathbb{N}} \in W$. We know that there exists $\beta \in \ell^p C^0(X,\xi)$ such that $\delta\beta = \overline{\theta}_1 - \overline{\theta}_2$. Observe that $\beta(0) = 0$ and for every $n \geq 1$

$$\beta(n) = \sum_{n\geq 1} (\overline{\theta}_1 - \overline{\theta}_2)([n-1,n]).$$

Since $\beta$ belongs to $\ell^p$ we have that $\lim_{n\to+\infty} \beta(n) = 0$ and then

$$\{(\overline{\theta}_1 - \overline{\theta}_2)([n,n+1])\}_{n\in\mathbb{N}} \in W.$$

The linearity and surjectivity of $g$ is clear.

Observe that $W \subset \ell^1(\mathbb{N}) \subset \ell^p(\mathbb{N})$ and the inclusion is strict in both cases. This and Claim 3 imply that

$$\dim(\ell^p H^1(X,\xi)) \geq \dim(\ell^p(\mathbb{N})/W) > 1 = \dim(\ell^p H^1(X,\xi_0)).$$

## 3.2 Equivalence between simplicial and de Rham relative $L^p$-cohomology

We say that a Riemannian manifold has bounded geometry if:

- its sectional curvature is bounded from below and above, and
- it has positive injectivity radius.

We refer to [Do92, GHL90] for more information about these two conditions.

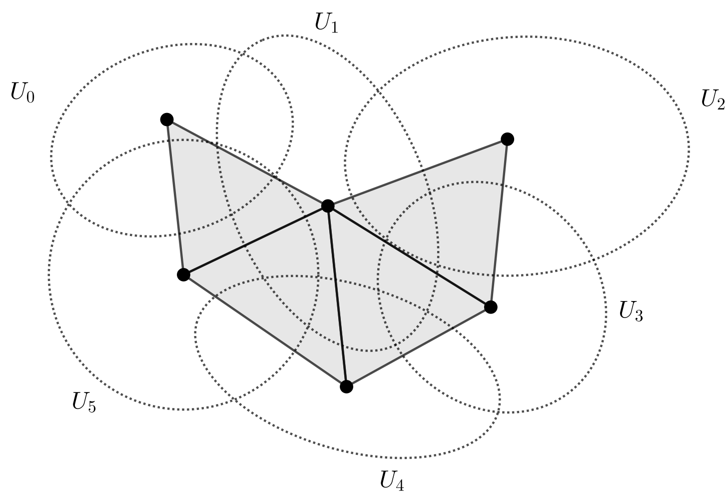
Let $M$ be a complete and Gromov-hyperbolic Riemannian manifold with bounded geometry, and $\xi$ a point in $\partial M$. Consider on $M$ a uniformly locally finite open covering $\mathcal{U}$ such that all non-empty intersections $U_1 \cap \ldots \cap U_k$, with $U_1, \ldots, U_k \in \mathcal{U}$, are bi-Lipschitz diffeomorphic to the unit ball in $\mathbb{R}^n$ ($n = \dim(M)$) with uniform Lipschitz constant. We say that the covering $\mathcal{U}$ is *uniformly locally finite* if there exists a constant $C \geq 1$ such that every point $x \in M$ belongs to at most $C$ elements of $\mathcal{U}$. Such a covering can be constructed using a triangulation of $M$ such that every simplex is uniformly bi-Lipschitz homeomorphic to the standard Euclidean simplex of the same dimension. For every vertex we consider $U(v)$ the interior of the union of all simplices containing $v$. Then we can define $\mathcal{U}$ as the collection of sets $U(v)$. In [Att94] it is shown how to construct a triangulation with this property in the case of bounded geometry. Another possibility is to consider the nerve of the covering constructed in [Gen14, Property 4.6.11].

For each $\ell \in \mathbb{N}$ we consider the set

$$\mathcal{U}_\ell = \{U_0 \cap \ldots \cap U_\ell \neq \emptyset : U_i \in \mathcal{U} \text{ for all } i = 0, \ldots, \ell\}.$$

Let $X_M$ be the nerve of the pair $(M,\mathcal{U})$, this is the simplicial complex such that:

- There is an $\ell$-simplex with vertices $U_0, \ldots, U_\ell$ if $U_0 \cap \ldots \cap U_\ell \in \mathcal{U}_\ell$. Thus we identify $X_M^\ell$ with $\mathcal{U}_\ell$.
- Every simplex is isometric to the standard Euclidean simplex of the same dimension.




Observe that $X_M$ is quasi-isometric to $M$. Moreover, there is a family of quesi-isometries $F : X_M \to M$ verifying $F(U) \in U$ for every vertex $U \in \mathcal{U}$, that we call *canonical quasi-isometries*. These canonical quasi-isometries are all at bounded uniform distance from each other; thus they represent an unique element of $QI(X_M, M)$, and therefore they induce the same map on the boundary. Denote by $\overline{\xi} \in X_M$ the point corresponding to $\xi$ by a canonical quasi-isometry. We say that $(X_M, \overline{\xi})$ is a *simplicial pair corresponding* to $(M, \xi)$. By construction, if $M$ is uniformly contractible, then so is $X_M$.

The strategy to prove Theorem 1.2.3 is to apply Lemma 2.2.1 to some convenient bicomplex. Before that we have to prove a pair of lemmas that will be used in the proof.

**Lemma 3.2.1** (Lemma 8 in [Pan95])**.** *Let $B$ be the unit ball in $\mathbb{R}^n$, then the cochain complex $(L^pC^*(B), d)$ retracts to the subcomplex $(\mathbb{R} \to 0 \to 0 \to \ldots)$.*

*Proof.* Fix $x \in B$. Suppose that $\chi : \Omega^k(B) \to \Omega^{k-1}(B)$ is defined for all $k \geq 1$ so that for every $(k-1)$-simplex $\tau \subset B$, we have

$$\int_\tau \chi(\omega) = \int_{C_\tau} \omega$$

for every differential $k$-form $\omega$. The cone $C_\tau$ is defined as follows: If $\tau = (x_0, \ldots, x_{k-1})$, then $C_\tau = (x, x_0, \ldots, x_{k-1})$. The function $\chi$ will depend on $x$, we write $\chi_x = \chi$ if necessary.

Claim:

$$\chi d + d\chi = \mathrm{Id}. \tag{3.2}$$

We take $\sigma$ a $k$-simplex in $B$ and $\omega \in \Omega^k(B)$, then

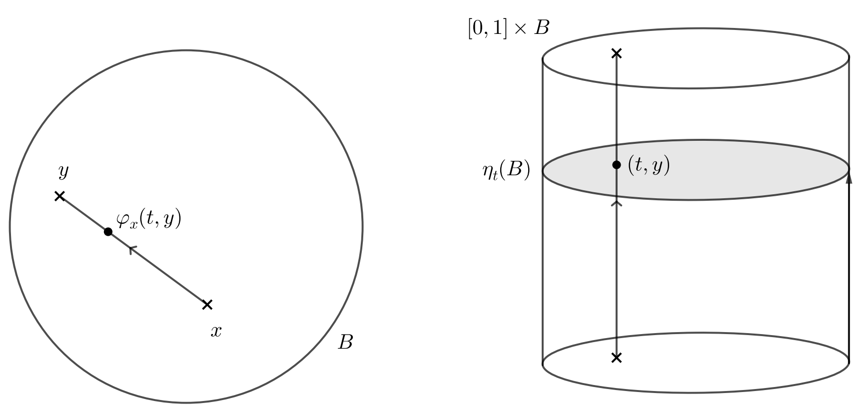
$$\int_\sigma \chi(d\omega) = \int_{C_\sigma} d\omega = \int_{\partial C_\sigma} \omega,$$

where the last equality comes from Stokes theorem. If $\partial\sigma = \tau_0 + \ldots + \tau_k$, we have

$$\begin{aligned}\int_\sigma \chi(d\omega) &= \int_\sigma \omega - \sum_{i=0}^{k} \int_{C_{\tau_i}} \omega = \int_\sigma \omega - \sum_{i=0}^{k} \int_{\tau_i} \chi(\omega) \\ &= \int_\sigma \omega - \int_{\partial\sigma} \chi(\omega) = \int_\sigma \omega - \int_\sigma d\chi(\omega).\end{aligned}$$

Since the equality holds for every $k$-simplex we conclude (3.2) (see for example [Whi57, Chapter IV]).

For $x \in B$ we consider $\varphi = \varphi_x : [0,1] \times B \to B$, $\varphi_x(t,y) = ty + (1-t)x$ and $\eta_t : B \to [0,1] \times B$, $\eta_t(y) = (t,y)$.



We look for an explicit expression for $\chi(\omega)$:

$$\int_\sigma \chi(\omega) = \int_{C_\sigma} \omega = \int_{\varphi([0,1]\times\sigma)} \omega = \int_{[0,1]\times\sigma} \varphi^*\omega = \int_\sigma \int_0^1 \eta_s^*(\iota_{\frac{\partial}{\partial t}} \varphi^*\omega) ds.$$

In the last equality we use Lemma 2.3.4. We conclude that

$$\chi(\omega) = \int_0^1 \eta_t^*(\iota_{\frac{\partial}{\partial t}} \varphi^*\omega) ds.$$

The family of $k$-form $(x,t) \mapsto \eta_t^*(\iota_{\frac{\partial}{\partial t}} \varphi^*\omega)$ satisfies the condition $(i)$ of Lemma 2.3.2 because it is smooth in both variables and the interval $[0,1]$ is compact; thus $\chi$ is smooth. By definition and the claim it satisfies equality (3.2). Observe that if $\omega$ is closed, then $\chi(\omega)$ is a primitive of $\omega$, so it is enough to prove the classic Poincaré's lemma. However, in our case we need an $L^p$-integrable primitive, so we take a convenient average. Define

$$h(\omega) = \frac{1}{\mathrm{Vol}\left(\frac{1}{2}B\right)} \int_{\frac{1}{2}B} \chi_x(\omega) dx,$$

where $\frac{1}{2}B = B\left(0, \frac{1}{2}\right)$.

Since $(x,y) \mapsto \chi_x(\omega)_y$ is smooth in both variables we can use again Lemma 2.3.2 to show that $h$ belongs to $\Omega^k(B)$. Notice that this works because we take the integral on a ball with closure included in $B$. Moreover, the derivative of $h$ is

$$dh(\omega) = \frac{1}{\mathrm{Vol}\left(\frac{1}{2}B\right)} \int_{\frac{1}{2}B} d\chi_x(\omega) dx.$$

Using (3.2) we have

$$dh(\omega) + h(d\omega) = \omega \tag{3.3}$$

for all $\omega \in L^p\Omega^k(B)$ with $k \geq 1$.

We want to prove that $h$ is well-defined from $L^p\Omega^k(B)$ to $L^p\Omega^{k-1}(B)$ and that it is continuous. To this end we first bound $|\chi_x(\omega)|_y$ for $y \in B$ and $\omega \in \Omega^k(B)$. Since $\iota_{\frac{\partial}{\partial t}} \varphi^*\omega$ is a form on $[0,1] \times B$ that is zero in the direction of $\frac{\partial}{\partial t}$, we have for all $t \in (0,1)$ and $y \in B$,

$$|\eta_t^*(\iota_{\frac{\partial}{\partial t}} \varphi^*\omega)|_y = |\iota_{\frac{\partial}{\partial t}} \varphi^*\omega|_{(t,y)}.$$

Then we can compute

$$\begin{aligned}
|\iota_{\frac{\partial}{\partial t}} \varphi^*\omega|_{(t,y)} &= \sup\{|\iota_{\frac{\partial}{\partial t}} \varphi^*\omega_{(t,y)}(v_1, \ldots, v_{k-1})| : \|v_1\| = \cdots = \|v_{k-1}\| = 1\} \\
&= \sup\left\{\left|\varphi^*\omega_{(t,y)}\left(\tfrac{\partial}{\partial t}, v_1, \ldots, v_{k-1}\right)\right| : \|v_1\| = \cdots = \|v_{k-1}\| = 1\right\} \\
&= \sup\{|\omega_{\varphi(t,y)}(y-x, tv_1, \ldots, tv_{k-1})| : \|v_1\| = \cdots = \|v_{k-1}\| = 1\} \\
&\leq t^{k-1}|y-x||\omega|_{\varphi(t,y)}.
\end{aligned}$$

From this and the assumption that $t \in (0,1)$ we get

$$|\chi(\omega)|_y \leq \int_0^1 |y-x||\omega|_{\varphi(t,y)} ds. \tag{3.4}$$

Consider the function $u : \mathbb{R}^n \to \mathbb{R}$ defined by $u(z) = |\omega|_z$ if $z \in B$ and $u(z) = 0$ in the other case. Using (3.4) we have

$$\mathrm{Vol}\left(\tfrac{1}{2}B\right)|h(\omega)|_y \leq \int_{\frac{1}{2}B}\int_0^1 |y-x|u(ty+(1-t)x)dtdx.$$

We write $z = ty + (1-t)x$, then

$$\begin{aligned}
\mathrm{Vol}(\tfrac{1}{2}B)|h(\omega)|_y &\leq \int_{B(ty,1-t)}\int_0^1 |z-y|u(z)(1-t)^{-n-1}dtdz \\
&\leq \int_{B(y,2)}\int_0^1 \mathbb{1}_{B(ty,1-t)}(z)|z-y|u(z)(1-t)^{-n-1}dtdz \\
&= \int_{B(y,2)} |z-y|u(z)\left(\int_0^1 \mathbb{1}_{B(ty,1-t)}(z)(1-t)^{-n-1}dt\right)dz.
\end{aligned}$$

Observe that $\mathbb{1}_{B(ty,1-t)}(z) = 1$ implies that $|z-y| \leq 2(1-t)$. Then we have

$$\int_0^1 \mathbb{1}_{B(ty,1-t)}(z)(1-t)^{-n-1}dt \leq \int_0^{1-\frac{1}{2}|z-y|}(1-t)^{-n-1}dt = \int_{\frac{1}{2}|z-y|}^1 r^{-n-1}dr \preceq \frac{1}{|z-y|^n}.$$

The notation $f \preceq g$ means that there exists a constant $C > 0$ such that $f \leq Cg$. This implies

$$\mathrm{Vol}(\tfrac{1}{2}B)|h(\omega)|_y \preceq \int_{B(y,2)} |z-y|^{1-n}u(z)dz.$$

Using that $\int_{B(y,2)}|z-y|^{1-n}dz$ is finite and Jensen's inequality we obtain

$$|h(\omega)|_y^p \preceq \int_{B(y,2)} |z-y|^{1-n}u(z)^p dz.$$

Therefore

$$\begin{aligned}
\|h(\omega)\|_{L^p}^p = \int_B |h(\omega)|_y^p dy &\preceq \int_B\int_{B(y,2)} |z-y|^{1-n}u(z)^p dzdy \\
&\preceq \int_{B(0,3)} u(z)^p\left(\int_B \frac{dy}{|z-y|^{n-1}}\right)dz \preceq \|\omega\|_{L^p}^p.
\end{aligned}$$

Using the identity $dh(\omega) = \omega - h(d\omega)$ we have

$$\|dh(\omega)\|_{L^p} \leq \|\omega\|_{L^p} + \|h(d\omega)\|_{L^p} \preceq |\omega|_{L^p}$$

We conclude that $h$ is well-defined and bounded for $k \geq 1$.

If $\omega = df$ for certain function $f$ we observe that

$$\eta_t^*(\iota_{\frac{\partial}{\partial t}\varphi_x^* df})(y) = df_{\varphi_x(t,y)}(y - x) = (f \circ \gamma)'(t),$$

where $\gamma$ is the curve $\gamma(t) = \varphi_x(t, y)$. Then $\chi_x(df)(y) = f(y) - f(x)$, from which we get

$$h(df) = f - \frac{1}{\mathrm{Vol}(\frac{1}{2}B)} \int_{\frac{1}{2}B} f.$$

We define $h : L^p\Omega^0(B) \to L^p\Omega^{-1}(B) = \mathbb{R}$ by

$$h(f) = \frac{1}{\mathrm{Vol}(\frac{1}{2}B)} \int_{\frac{1}{2}B} f,$$

which is crearly continuous because $\frac{1}{2}B$ has finite Lebesgue measure. Therefore the identity (3.3) is true for every $k \geq 1$ and $\omega \in L^p\Omega^k(B)$, and $h$ is continuous in all degrees.

Note that, since $h$ is bounded, then it can be extended continuously to $L^pC^k(B)$ for every $k \geq 0$. The equality (3.3) is also true for every $\omega \in L^pC^k(B)$, then it is the retraction we wanted. $\square$

**Lemma 3.2.2.** *Let $f : M \to N$ be a bi-Lipschitz diffeomorphism, where $M$ and $N$ are Riemannian manifolds. Then the pull-back $f^* : L^pC^k(N) \to L^pC^k(M)$ is well-defined and continuous. Furthermore, the operator norm of $f^*$ is bounded depending on the Lipschitz constant of $f$, $n = \dim(M)$, $p$ and $k$.*

*Proof.* Suppose that $L$ is the Lipschitz constant of $f$. Let $\omega \in L^pC^k(N)$, by Remark 1.2.2 we can see $\omega$ and its derivative as elements of $L^p(N, \Lambda^k)$, then

$$\begin{aligned}
|f^*\omega|_x &= \inf\left\{\left|f^*\omega_x\left(\frac{v_1}{\|v_1\|_x}, \ldots, \frac{v_k}{\|v_k\|_x}\right)\right| : v_1, \ldots, v_k \in T_xM\right\} \\
&= \inf\left\{\left|\omega_{f(x)}\left(\frac{d_xf(v_1)}{\|v_1\|_x}, \ldots, \frac{d_xf(v_k)}{\|v_k\|_x}\right)\right| : v_1, \ldots, v_k \in T_xM\right\} \\
&\leq L^k \inf\left\{\left|\omega_{f(x)}\left(\frac{w_1}{\|w_1\|_{f(x)}}, \ldots, \frac{w_k}{\|w_k\|_{f(x)}}\right)\right| : w_1, \ldots, w_k \in T_{f(x)}N\right\} \\
&= L^k|\omega|_{f(x)}.
\end{aligned}$$

Then

$$\begin{aligned}
\|f^*\omega\|_{L^p}^p &= \int_M |f^*\omega|_x^p dV_M(x) \leq \int_M L^{pk}|\omega|_{f(x)}^p L^n |Jac_x(f)| dV_M(x) \\
&= L^{n+pk} \int_N |\omega|_y^p dV_N(y) = L^{n+pk}\|\omega\|_{L^p}^p.
\end{aligned}$$

Using that the pull-back commutes with the derivative, the same argument shows that $\|df^*\omega\|_{L^p}^p = \|f^*d\omega\|_{L^p}^p \le L^{n+p(k+1)}\|dw\|_{L^p}^p$. $\square$

*Proof of Theorem 1.2.3.* We define the bicomplex $(C_\xi^{*,*}, d', d'')$ as follows:

First consider

$$C^{k,\ell} = \left\{ \omega \in \prod_{U\in\mathcal{U}_\ell} L^pC^k(U) : \sum_{U\in\mathcal{U}_\ell} \|\omega_U\|_{L^p}^p + \|d\omega_U\|_{L^p}^p < +\infty \right\},$$

with the norm

$$\|\omega\| = \left(\sum_{U\in\mathcal{U}_\ell} \|\omega_U\|_{L^p}^p\right)^{\frac{1}{p}} + \left(\sum_{U\in\mathcal{U}_\ell} \|d\omega_U\|_{L^p}^p\right)^{\frac{1}{p}}.$$

Then $C_\xi^{k,\ell}$ is the subspace of all elements $\omega \in C^{k,\ell}$ for which there exists $V$ a neigborhood of $\xi$ in $\overline{M}$ such that $\omega_U = 0$ for all $U \subset V$. We define the derivatives $d' : C_\xi^{k,\ell} \to C_\xi^{k+1,\ell}$ and $d'' : C_\xi^{k,\ell} \to C_\xi^{k,\ell+1}$:

- If $\omega \in C_\xi^{k,\ell}$, then $(d'\omega)_U = (-1)^\ell d\omega_U$.
- If $\omega \in C_\xi^{k,\ell}$ and $W \in \mathcal{U}_{\ell+1}$, $W = U_0 \cap \ldots \cap U_{\ell+1}$, then

$$(d''\omega)_W = \sum_{i=0}^{\ell+1} (-1)^i (\omega_{U_0\cap\ldots\cap U_{i-1}\cap U_{i+1}\cap\ldots\cap U_{\ell+1}})|_W.$$

It is easy to show that $d'$ and $d''$ are continuous and satisfy $d' \circ d'' + d'' \circ d' = 0$.

Observe that the elements of $\operatorname{Ker} d'|_{C_\xi^{0,\ell}}$ are the functions $g \in \prod_{U\in\mathcal{U}_\ell} L^pC^0(U)$ satisfying the following conditions:

- There exists $V \subset \overline{M}$ a neighborhood of $\xi$ such that $g_U = 0$ if $U \subset V$.
- $dg_U = 0$ for all $U \in \mathcal{U}_\ell$, then $g_U$ is essentially constant.
- $\sum_{U\in\mathcal{U}_\ell} \int_U |g_U|^p dV < +\infty$.

Using the construction of $X_M$ and the fact that $U$ is bi-Lipschitz diffeomorphic (with uniform Lipschitz constant) to the unit ball in $\mathbb{R}^n$ we have that $\operatorname{Ker} d'|_{C_\xi^{0,\ell}}$ is isomorphic to $\ell^pC^\ell(X, \bar{\xi})$. Indeed, the map

$$\ell^pC^\ell(X, \bar{\xi}) \to \operatorname{Ker} d'|_{C_\xi^{0,\ell}},\ \theta \mapsto g^\theta,$$

where $g_U^\theta$ is constant $\theta(U)$, satisfy

$$\inf\{\mathrm{Vol}(U) : U \in \mathcal{U}\}^{\frac{1}{p}} \|\theta\|_{\ell^p} \le \|g^\theta\| \le \sup\{\mathrm{Vol}(U) : U \in \mathcal{U}\}^{\frac{1}{p}} \|\theta\|_{\ell^p}.$$

Observe that $d''$ coincides with the derivative $\delta$ on $\ell^pC^\ell(X,\bar{\xi})$ via this identification, i.e. $g^{\delta\theta} = d''g^\theta$.

On the other hand the elements of $\operatorname{Ker} d''|_{C_\xi^{k,0}}$ are of the form $\omega = \{\omega_U\}_{U\in\mathcal{U}}$ with

$$\omega_U|_{U\cap U'} = \omega_{U'}|_{U\cap U'} \text{ a.e. if } U\cap U' \neq \emptyset.$$

We can take a $k$-form $\tilde{\omega}$ in $L^pC^k(M)$ such that $\tilde{\omega}|_U = \omega_U$ a.e. for all $U \in \mathcal{U}$. This $k$-form is zero in some neighborhood of $\xi$, then the identity is an isomorphism between $\operatorname{Ker} d''|_{C_\xi^{k,0}}$ and $L^pC^k(M,\xi)$. It is clear that $d' = d$ in this case.

$$\begin{tikzcd}
 & & L^pC^0(M,\xi) \arrow[r, "d"] \arrow[d, "\sim"] & L^pC^1(M,\xi) \arrow[r, "d"] \arrow[d, "\sim"] & \cdots \\
 & & \operatorname{Ker} d''|_{C_\xi^{0,0}} \arrow[r, "d'"] \arrow[d, hook] & \operatorname{Ker} d''|_{C_\xi^{1,0}} \arrow[r, "d'"] \arrow[d, hook] & \cdots \\
\ell^pC^0(X_M,\bar{\xi}) \arrow[r, "\sim"] \arrow[d, "\delta"] & \operatorname{Ker} d'|_{C_\xi^{0,0}} \arrow[r, hook] \arrow[d, "d''"] & C_\xi^{0,0} \arrow[r, "d'"] \arrow[d, "d''"] & C_\xi^{1,0} \arrow[r, "d'"] \arrow[d, "d''"] & \cdots \\
\ell^pC^1(X_M,\bar{\xi}) \arrow[r, "\sim"] \arrow[d, "\delta"] & \operatorname{Ker} d'|_{C_\xi^{0,1}} \arrow[r, hook] \arrow[d, "d''"] & C_\xi^{0,1} \arrow[r, "d'"] \arrow[d, "d''"] & C_\xi^{1,1} \arrow[r, "d'"] \arrow[d, "d''"] & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

<u>Claim 1</u>: For a fixed $\ell$, $(C_\xi^{*,\ell}, d')$ retracts to $(\operatorname{Ker} d'|_{C_\xi^{0,\ell}} \to 0 \to 0 \to \cdots)$.

Lemma 3.2.1 implies that there exists a family of bounded maps $h : L^pC^k(B) \to L^pC^{k-1}(B)$ such that $h\circ d + d\circ h = \mathrm{Id}$. We denote $L^pC^{-1}(B) = \mathbb{R}$ and $d : L^pC^{-1}(B) \to L^pC^0(B)$ the inclusion. Consider for every $U \in \mathcal{U}_\ell$ a smooth bi-Lipschitz function $f_U : U \to B$ with constant $K$ (which does not depend on $U$). Then we define $H : C_\xi^{k,\ell} \to C_\xi^{k-1,\ell}$ by

$$(H\omega)_U = f_U^* h((f_U^{-1})^*\omega_U).$$

We write $C_\xi^{-1,\ell} := \operatorname{Ker} d'|_{C_\xi^{0,\ell}}$. Using Lemma 3.2.2 and the definition of $h$ we can see that $H$ defines the retraction we wanted. In particular it is bounded.

<u>Claim 2</u>: For a fixed $k$, $(C_\xi^{k,*}, d'')$ retracts to $(\operatorname{Ker} d''|_{C_\xi^{k,0}} \to 0 \to 0 \to \cdots)$.

We have to construct a family of bounded linear maps $P : C_\xi^{k,\ell} \to C_\xi^{k,\ell-1}$ ($\ell \geq 0$) such that $P\circ d'' + d''\circ P = \mathrm{Id}$, where $C_\xi^{k,-1} = \operatorname{Ker} d''|_{C_\xi^{k,0}}$ and $d'' : C_\xi^{k,-1} \to C_\xi^{k,0}$ is the inclusion.

Consider $\{\eta_U\}_{U\in\mathcal{U}}$ a partition of unity with respect to $\mathcal{U}$. If $\ell \geq 1$ and $\omega \in C_\xi^{k,\ell}$, then we define

$$(P\omega)_V = \sum_{U\in\mathcal{U}} \eta_U \omega_{U\cap V}$$

for all $V \in \mathcal{U}_{\ell-1}$. For $\omega \in C_\xi^{k,0}$ and $V \in \mathcal{U}$ we put

$$(P\omega)_V = \sum_{U\in\mathcal{U}} \eta_U \omega_U|_V.$$

A direct calculation shows that $P$ is as we wanted.

Finally, aplying Lemma 2.2.1 we obtain that $(D^*, \delta)$ is homotopically equivalent to $(\operatorname{Ker} d'|_{C_\xi^{0,*}}, d'')$ and $(\operatorname{Ker} d''|_{C_\xi^{*,0}}, d')$. The proof ends using the above identifications. $\square$

Observe that in the previous proof we can consider the bicomplex given by the elements of

$$\tilde{C}^{k,\ell} = \left\{ \omega \in \prod_{U\in\mathcal{U}_\ell} L^p\Omega^p(U) : \sum_{U\in\mathcal{U}_\ell} \|\omega_U\|_p^p + \|d\omega_U\|_p^p < +\infty \right\}$$

which vanish on a neighborhood of $\xi$. Following the same arguments (which involves the observation that Lemma 3.2.1 is true also for the complex ($L^p\Omega^*(B)$,d)) we can prove the homotopy equivalence between the cochain complexes $(\ell^p C^*(X_M, \bar{\xi}), \delta)$ and $(L^p\Omega^*(M, \xi), d)$, and as a consequence Theorem 1.2.4.

## 3.3 Some duality ideas

In [GKS86] and [GT10] the following fact is proved: If $M$ is a complete and orientable $n$-dimensional Riemannian manifold, then for every $p \in (1, +\infty)$ and $k = 0, \ldots, n$, the dual space of $L^p\overline{H}^k(M)$ is isometric to $L^q\overline{H}^{n-k}(M)$, where $\frac{1}{p} + \frac{1}{q} = 1$. The isometry is induced by the pairing $\langle\ ,\ \rangle : L^p(M, \Lambda^k) \times L^q(M, \Lambda^{n-k}) \to \mathbb{R}$ defined by

$$\langle \omega, \beta \rangle = \int_M \omega \wedge \beta, \tag{3.5}$$

wich is well-defined by Hölder's inequality. The proof uses that $L^p(M, \Lambda^k)$ and $L^q(M, \Lambda^{n-k})$ are Banach spaces. The relative case is a very diferent context, however it makes sense to ask the following question: What would be the natural pairing for $L^p\Omega^k(M, \xi)$ (or $L^pC^k(M, \xi)$) instead of $L^p(M, \Lambda^k)$?

The answer seems to be related to the idea of local cohomology, which can be found in [Car16]. Let us see the following definition: Consider $M$ a complete and orientable Gromov-hyperbolic Riemannian manifold and $\xi$ a point in $\partial M$. A differential $m$-form $\beta$

on $M$ is *locally $L^q$-integrable with respect to* $\xi$ if for every $V \subset \overline{M}$, a closed neighborhood of $\xi$, we have that

$$\|\beta\|_{L^q, M\setminus V} = \left(\int_{M\setminus V} |\beta|_x^q dx\right)^{\frac{1}{q}} < +\infty.$$

Then we define $L^q_{loc}\Omega^m(M,\xi)$ as the space of all differential $m$-forms which are locally $L^q$-integrable with respect to $\xi \in \partial M$. Observe that Hölder's inequality implies that the bi-linear pairing

$$\langle\ ,\ \rangle : L^p\Omega^k(M,\xi) \times L^q_{loc}\Omega^{n-k}(M,\xi) \to \mathbb{R} \tag{3.6}$$

is well-defined by the expression (3.5) if $\frac{1}{p} + \frac{1}{q} = 1$. This allows to consider the induced linear transformations $\mu_\omega : L^q_{loc}\Omega^{n-k}(M,\xi) \to \mathbb{R}$, $\mu_\omega = \langle \omega, \cdot\rangle$ and $\nu_\beta : L^p\Omega^k(M,\xi) \to \mathbb{R}$, $\nu_\beta = \langle \cdot, \beta\rangle$.

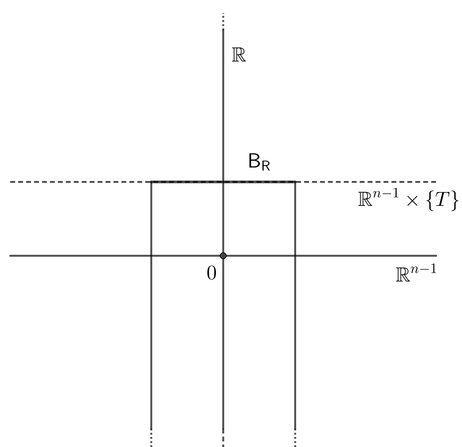
## 3.4 An application to Heintze groups

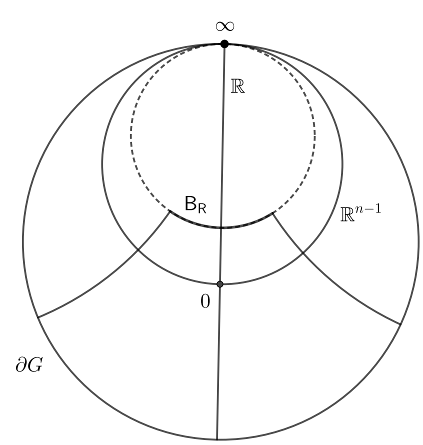
Let $G = \mathbb{R}^{n-1} \rtimes_\alpha \mathbb{R}$ be a purely real Heintze group where $\alpha$ has positive eigenvalues $\lambda_1 \leq \cdots \leq \lambda_{n-1}$. The product on $G$ is given by

$$(x,t)\cdot(y,s) = (x + e^{t\alpha}y, t+s).$$

We denote by $L_{(x,t)}$ and $R_{(x,t)}$ the left and right translations by $(x,t)$ on $G$.

Observe that a neighborhood system for the point $\infty \in \partial G$ is given by the compactification in $\overline{G}$ of sets of the form $G \setminus (B_R \times [T, -\infty))$, where $B_R = B(0,R) \in \mathbb{R}^{n-1}$ for some positive number $R$, and $T \in \mathbb{R}$. This will be important to work with the $L^p$-cohomology relative to $\infty$.



If $\langle\ ,\ \rangle_0$ is an inner product on $T_0G$ such that the factors $\mathbb{R}^{n-1}$ and $\mathbb{R}$ are orthogonal, then it determines an unique left-invariant metric on $G$ defined by

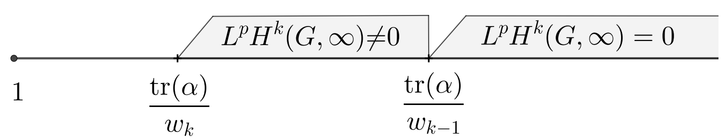
$$\begin{aligned}\langle (v_1, v_2), (w_1, w_2)\rangle_{(x,t)} &= \langle (d_0L_{(x,t)})^{-1}(v_1, v_2), (d_0L_{(x,t)})^{-1}(w_1, w_2)\rangle_0 \\ &= \langle e^{-t\alpha}v_1, e^{-t\alpha}w_1\rangle_0 + \lambda v_2 w_2,\end{aligned}$$

where $v_1, w_1 \in \mathbb{R}^{n-1}$, $v_2, w_2 \in \mathbb{R}$ and $\lambda$ is a fixed positive real number. In particular, if $v$ is a *horizontal vector* in $T_{(x,t)}G$ (i.e. $v = (v_1, 0)$), then the norm associated to $\langle\ ,\ \rangle_{(x,t)}$ of $v$ is

$$\|v\|_{(x,t)} = \|e^{-t\alpha}v\|_0.$$

For $k = 1, \ldots, n-1$ consider the number $w_k = w_k(\alpha) = \lambda_1 + \cdots + \lambda_k$. The aim of this section is to prove the following result:

**Theorem 3.4.1.** *Let $k = 2, \ldots, n-1$, then $L^pH^k(G, \infty) = 0$ for all $p > \frac{\operatorname{tr}(\alpha)}{w_{k-1}}$ and $L^pH^k(G, \infty) \neq 0$ for all $p \in \left(\frac{\operatorname{tr}(\alpha)}{w_k}, \frac{\operatorname{tr}(\alpha)}{w_{k-1}}\right]$.*




The following lemma is proved more generally in [Cor18, Lemma 6.D.1].

**Lemma 3.4.2.** *Consider two Heintze groups $G_1 = N_1 \rtimes_{\alpha_1} \mathbb{R}$ and $G_2 = N_2 \rtimes_{\alpha_2} \mathbb{R}$. If $G_1$ and $G_2$ are quasi-isometric, then there exists a quasi-isometry $F : G_1 \to G_2$ such that $F(\infty) = \infty$.*

*Proof.* Since $N_i$ $(i = 1, 2)$ acts on $G_i$ by isometries, there are two possibilities:

- $QI(G_i)$ acts transitively on $\partial G_i$, or
- $\infty$ is fixed by $QI(G_i)$.

If $G_1$ is in the first case, then $G_2$ too and every quasi-isometry between $G_1$ and $G_2$ carries $\infty$ to $\infty$. In the second case it is enough to take a quasi-isometry $F : G_1 \to G_2$ and then $H \in QI(G_2)$ such that $H(F(\infty)) = \infty$. The composition $H \circ F$ is the quasi-isometry we wanted. $\square$

Combining the previous lemma with Theorem 3.4.1 and Corollary 1.2.5 we deduce:

**Corollary 3.4.3.** *Let $G_1 = \mathbb{R}^{n-1} \rtimes_{\alpha_1} \mathbb{R}$ and $G_2 = \mathbb{R}^{n-1} \rtimes_{\alpha_2} \mathbb{R}$ be two purely real Heintze groups. If $G_1$ and $G_2$ are quasi-isometric, then for all $k = 1, \ldots, n-1$, we have $\frac{tr(\alpha_1)}{w_k(\alpha_1)} = \frac{tr(\alpha_2)}{w_k(\alpha_2)}$.*

Note that Theorem 1.2.7 follows as a direct consequence of Corollary 3.4.3, more precisely we have that $\alpha_1$ and $\frac{\mathrm{tr}(\alpha_1)}{\mathrm{tr}(\alpha_2)}\alpha_2$ have the same eigenvalues.

As we saw in Theorem 1.2.4, we can restrict to differential forms. In this section we use the notation $L^pH^k(G,\infty)$ to mean the cohomology spaces of the cochain complex $(L^p\Omega^*(G,\infty), d)$.

We start with the diagonalizable case because it is easier from the technical point of view and it is enough to show the main ideas of the proof of Theorem 3.4.1.

### 3.4.1 Diagonalizable case

Let us suppose that $\alpha$ is diagonalizable. The Lie bracket in Lie$(G)$ is defined by

$$[(X,T),(Y,S)] = T\alpha(Y) - S\alpha(X),$$

where $X \in \mathbb{R}^{n-1}$ and $T \in \mathbb{R}$. Note that if $\beta = P^{-1}\alpha P$ with $P \in GL(\mathbb{R}, n)$, then

$$\mathrm{Lie}(\mathbb{R}^{n-1} \rtimes_\beta \mathbb{R}) \to \mathrm{Lie}(\mathbb{R}^{n-1} \rtimes_\alpha \mathbb{R}),\ (X,T) \mapsto (PX, T)$$

defines an isomorphism of Lie algebras. This implies that both Heintze groups are isomorphic and then quasi-isometric. So we can suppose that $\alpha$ is diagonal with the eigenvalues in increasing order on the diagonal.

Denote by $dx$ and $dt$ the Lebesgue measure on $\mathbb{R}^{n-1}$ and $\mathbb{R}$ respectively. Consider $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbb{R}^n$ and $\{e_1^*, \ldots, e_n^*\}$ its dual basis. The differential 1-form $dx_i$ on $G$ ($i = 1, \ldots, n$) is defined by $(dx_i)_{(x,t)} = e_i^*$. We will be a bit ambiguous and also use the notation $dt = dx_n$. The left-invariant metric we consider in $G$ is the one generated by the Euclidean inner product on $T_0G = \mathbb{R}^n$.

Left translations acts on 1-forms in the following way:

$$L^*_{(x,t)}\left(\sum_{i=1}^{n-1} a_i\ dx_i + a_n\ dt\right) = \sum_{i=1}^{n-1} e^{t\lambda_i}(a_i \circ L_{(x,t)})\ dx_i + (a_n \circ L_{(x,t)})\ dt.$$

In particular $L^*_{(x,t)}dx_i = e^{t\lambda_i}dx_i$ for all $i = 1, \ldots, n-1$.

Observe that if $\omega$ is a $k$-form on $G$, then

$$|\omega|_{(x,t)} = |L^*_{(x,t)}\omega|_0$$

for all $(x,t) \in G$. Thus the operator norm is left-invariant.

**Lemma 3.4.4.** *(i)* $|dx_{i_1}\wedge\ldots\wedge dx_{i_k}|_{(x,t)} \asymp e^{t(\lambda_{i_1}+\ldots+\lambda_{i_k})}$ *for* $1 \le i_1 < \ldots < i_k \le n-1$.

*(ii) The volume form on* $G$ *is* $dV(x,t) = e^{t\mathrm{tr}(\alpha)}dx_1 \wedge \ldots \wedge dx_n$.

*Proof.* (i) On $\Lambda^k(T_0G)$ we consider the inner product $\langle\langle\ ,\ \rangle\rangle_0$ that makes the basis $\{e^*_{i_1} \wedge \ldots \wedge e^*_{i_k} : 1 \le i_1 < \ldots < i_k \le n\}$ orthonormal. On $\Lambda^k(T_{(x,t)}G)$ we define $\langle\langle\ ,\ \rangle\rangle_{(x,t)}$ such that for all $\beta, \gamma \in \Lambda^k(T_{(x,t)}G)$ we have

$$\langle\langle \beta, \gamma\rangle\rangle_{(x,t)} = \langle\langle L^*_{(x,t)}\beta, L^*_{(x,t)}\gamma\rangle\rangle_0.$$

This means that the inner product is left-invariant.

The left-invariant norm induced by this inner product is denoted by $[\ \ ]_{(x,t)}$. Since the operator norm $|\ \ |_{(x,t)}$ is also left-invariant, there exists a constant $C \ge 1$ independent of the point $(x,t) \in G$ such that,

$$C^{-1}|\ \ |_{(x,t)} \le [\ \ ]_{(x,t)} \le C|\ \ |_{(x,t)}.$$

As a consequence it is enough to prove $(i)$ for $[\ \ ]_{(x,t)}$:

$$\begin{aligned}
[dx_{i_1} \wedge \ldots \wedge dx_{i_k}]_{(x,t)} &= [L^*_{(x,t)}(dx_{i_1} \wedge \cdots \wedge dx_{i_k})]_0 \\
&= [(L^*_{(x,t)}dx_{i_1}) \wedge \cdots \wedge (L^*_{(x,t)}dx_{i_k})]_0 \\
&= [(e^{t\lambda_{i_1}}dx_{i_1}) \wedge \cdots \wedge (e^{t\lambda_{i_k}}dx_{i_k})]_0 \\
&= e^{t(\lambda_{i_1}+\cdots+\lambda_{i_k})}[dx_{i_1} \wedge \cdots \wedge dx_{i_k}]_0 \\
&= e^{t(\lambda_{i_1}+\cdots+\lambda_{i_k})}
\end{aligned}$$

(ii) Here it is enough to prove that $e^{t\mathrm{tr}(\alpha)}dx_1 \wedge \ldots \wedge dx_n(v_1, \ldots, v_n) = 1$ for some positive orthonormal basis $\{v_1, \ldots, v_n\}$ in $T_{(x,t)}G$, for example

$$\{e^{t\lambda_1}e_1, \ldots, e^{t\lambda_{n-1}}e_{n-1}, e_n\}.$$

□

Let $V$ be the vertical vector field defined by $V(x,t) = e_n$, and $\varphi_t(x,s) = (x, s+t)$ its associated flow. We say that a $k$-form $\omega$ is *horizontal* if $\iota_V\omega = 0$. Observe that if

$$\omega = \sum_{1\le i_1<\ldots<i_k\le n} a_{i_1,\ldots,i_k}dx_{i_1} \wedge \ldots \wedge dx_{i_k}, \tag{3.7}$$

then $\omega$ is horizontal if, and only if, all coefficients $a_{i_1,\ldots,i_{k-1},n}$ are zero.

**Lemma 3.4.5.** *If* $\omega$ *is a horizontal* $k$*-form, then for all* $x \in \mathbb{R}^{n-1}$*,* $s \in \mathbb{R}$ *and* $t \ge 1$ *we have*

$$|\varphi^*_t\omega|_{(x,s)} \preceq e^{-tw_k}|\omega|_{(x,s+t)}.$$

*Proof.* Suppose that $\omega$ is as in (3.7). Using the norm $[\ \ ]_{(x,t)}$ as in Lemma 3.4.4, we have

$$\begin{aligned}\frac{[\varphi_t^*\omega]^2_{(x,s)}}{[\omega]^2_{(x,s+t)}} &= \frac{\sum |a_{i_1,\ldots,i_k}(x,s+t)|^2[dx_{i_1}\wedge\ldots\wedge dx_{i_k}]^2_{(x,s)}}{\sum |a_{i_1,\ldots,i_k}(x,s+t)|^2[dx_{i_1}\wedge\ldots\wedge dx_{i_k}]^2_{(x,s+t)}}\\ &= \sum \frac{|a_{i_1,\ldots,i_k}(x,s+t)|^2 e^{2s(\lambda_{i_1}+\ldots+\lambda_{i_k})}}{|a_{i_1,\ldots,i_k}(x,s+t)|^2 e^{2(s+t)(\lambda_{i_1}+\ldots+\lambda_{i_k})}}\\ &= \sum e^{-t(\lambda_{i_1}+\ldots+\lambda_{i_k})} \preceq e^{-tw_k}.\end{aligned}$$

□

We prove now the first part of Theorem 3.4.1 following the idea of [Pan08, Proposition 10].

**Proposition 3.4.6.** *Let $k = 2,\ldots,n$, then $L^pH^k(G,\infty) = 0$ for all $p > \frac{\mathrm{tr}(\alpha)}{w_{k-1}}$.*

*Proof.* Take $\omega$ a closed form in $L^p\Omega^k(G,\infty)$. We want to construct an $L^p$-integrable differential $(k-1)$-form $\vartheta$ such that $d\vartheta = \omega$.

Set

$$\vartheta = -\int_0^{+\infty} \varphi_t^*\iota_V\omega\ dt. \tag{3.8}$$

Observe that, since $\omega$ vanishes on a neighborhood of $\infty$, we have the pointwise convergence of the above integral, so $\vartheta$ is well-defined as a $k$-form.

Since $\iota_V\omega$ is a horizontal form, by Lemma 3.4.5 we have that for all $(x,s)\in G$ and $t \geq 0$,

$$|\varphi_t^*\iota_V\omega|_{(x,s)} \leq Ce^{-tw_k}|\iota_V\omega|_{(x,s+t)},$$

for some constant $C$. Then

$$\begin{aligned}\|\varphi_t^*\iota_V\omega\|^p_{L^p} &= \int_G |\varphi_t^*\iota_V\omega|^p_{(x,s)}dV(x,s)\\ &\leq C\int_G e^{-tpw_k}|\iota_V\omega|^p_{(x,s+t)}e^{-s\mathrm{tr}(\alpha)}dxds\\ &= C\int_G e^{-t(pw_k-\mathrm{tr}(\alpha))}|\iota_V\omega|^p_{(x,s+t)}e^{-(s+t)\mathrm{tr}(\alpha)}dxds\\ &= C\int_G e^{-t(pw_k-\mathrm{tr}(\alpha))}|\iota_V\omega|^p_{(x,s+t)}dV(x,s+t)\\ &= Ce^{-t\epsilon}\|\iota_V\omega\|^p_{L^p},\end{aligned}$$

where $\epsilon = pw_k - \mathrm{tr}(\alpha) > 0$. It is easy to see that $|\iota_V\omega|_{(x,s)} \leq |\omega|_{(x,s)}$ for all $(x,s)\in G$, so $\|\varphi_t^*\iota_V\omega\|_{L^p} \leq Ce^{-t\epsilon}\|\omega\|_{L^p}$. This implies that the integral (3.8) converges in $L^p(M,\Lambda^{k-1})$. It is also clear that $\vartheta$ vanishes on a neighbourhood of $\infty$. We have to prove that it is smooth and $d\vartheta = \omega$.

We know that there exists $T \in \mathbb{R}$ such that $\iota_V \omega_{(x,s)} = 0$ for all $s \geq T$, then $\vartheta_{(x,s)}$ is an integral on a compact interval for every $(x,s) \in M$. Since $(x,s,t) \mapsto \varphi_t^* \iota_V \omega$ is smooth we can use Lemma 2.3.2 to see that $\vartheta$ is in $\Omega^{k-1}(M)$ and

$$d\vartheta = -\int_0^{+\infty} d(\varphi_t^* \iota_V \omega) dt$$

The Lie derivative of $\omega$ with respect to the vertical field $V$ is

$$L_V \omega = \left.\frac{d}{dt}\right|_{t=0} \varphi_t^* \omega.$$

Observe that $\frac{d}{dt}\varphi_t^* \omega = \varphi_t^* L_V \omega$. Then using the Cartan formula $L_V \omega = d\iota_V \omega + \iota_V d\omega$ (see for example [GHL90, Chapter I,Section A]) and that $\omega$ is closed, we obtain

$$\varphi_t^* \omega - \omega = \int_0^t \frac{d}{ds} \varphi_s^* \omega \; ds = \int_0^t \varphi_s^*(d\iota_V \omega + \iota_V d\omega) ds = \int_0^t d(\varphi_s^* \iota_V \omega) ds.$$

For every $(x,r) \in G$ we have

$$\omega_{(x,r)} = \lim_{t \to +\infty} \left( \varphi_t^* \omega_{(x,r)} - \int_0^t d(\varphi_s^* \iota_V \omega)_{(x,r)} ds \right).$$

The limit exists because the expression in brackets is constant for $t$ big enough. Then we conclude

$$\omega_{(x,r)} = -\int_0^{+\infty} d(\varphi_s^* \iota_V \omega)_{(x,r)} ds = d\vartheta_{(x,r)}$$

for all $(x,t) \in G$, which finishes the proof. □

**Proposition 3.4.7.** *For $k = 2, \ldots, n-1$ and $p \in \left( \frac{\operatorname{tr}(\alpha)}{w_k}, \frac{\operatorname{tr}(\alpha)}{w_{k-1}} \right)$ we have $L^p H^k(G, \infty) \neq 0$.*

*Proof.* We want to construct a closed differential $k$-form $\omega$ on $G$ which represents a non-zero class in $L^p$-cohomology relative to $\infty$. Remember that we are working with the complex $(L^p\Omega^*(G,\infty), d)$. The strategy is inspired by the duality ideas mentioned in Section 3.3, that is: we give a $(n-k)$-form $\beta \in \Omega_{\text{loc}}^{q,n-k}(G,\infty)$, with $\frac{1}{p} + \frac{1}{q} = 1$, such that

(a) $\nu_\beta(\omega) = \int_G \omega \wedge \beta \neq 0$, and

(b) $dL^p\Omega^{k-1}(G,\infty) \subset \operatorname{Ker} \nu_\beta$;

which shows that $\omega$ represents a non-zero element in $L^p H^k(G,\infty)$.

Consider two smooth functions $g : (-\infty, +\infty) \to [0,1]$ and $f : \mathbb{R}^{n-1} \to [0,1]$ such that:

- $\mathrm{supp}(f)$ is compact, and
- $g(t) = 0$ for all $t \geq 1$ and $g(t) = 1$ for all $t \leq 0$.

We define $\omega_{(x,t)} = d\left(f(x)g(t)\ dx_1 \wedge \ldots \wedge dx_{k-1}\right)$. Using triangular inequality we have

$$\|\omega\|_p \leq \|fg'\ dt \wedge dx_1 \wedge \cdots \wedge dx_{k-1}\|_{L^p} + \sum_{j=k}^{n-1} \left\| \frac{\partial f}{\partial x_j} g\ dx_j \wedge dx_1 \wedge \cdots \wedge dx_{k-1} \right\|_{L^p}.$$

Observe that the first term is finite because $fg'$ is smooth and has compact support. Then it is enough to show that for all $j = k, ..., n-1$ the form $\omega_j = \frac{\partial f}{\partial x_j} g\ dx_j \wedge dx_1 \wedge \cdots \wedge dx_{k-1}$ is in $L^p$:

$$\begin{aligned}
\|\omega_j\|_{L^p}^p &= \int_G \left| \frac{\partial f}{\partial x_j}(x)g(t)\ dx_j \wedge dx_1 \wedge \cdots \wedge dx_{k-1} \right|_{(x,t)}^p dV(x,t) \\
&= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{1} \left| \frac{\partial f}{\partial x_j}(x) \right|^p |g(t)|^p \left| dx_j \wedge dx_1 \wedge \cdots \wedge dx_{k-1} \right|_{(x,t)}^p e^{-t\mathrm{tr}(\alpha)} dt dx \\
&\preceq \left\| \frac{\partial f}{\partial x_j} \right\|_{L^p}^p \int_{-\infty}^{1} e^{t(p(w_{k-1}+\lambda_j)-\mathrm{tr}(\alpha))} dt.
\end{aligned}$$

So $\|\omega_j\|_{L^p} < +\infty$ if $p > \frac{\mathrm{tr}(\alpha)}{w_{k-1}+\lambda_j}$ for every $j = k, \ldots, n-1$, which implies that $\|\omega\|_{L^p} < +\infty$ if $p > \frac{\mathrm{tr}(\alpha)}{w_k}$.

Define $\beta = dx_k \wedge \ldots \wedge dx_{n-1}$. To prove that $\beta$ is in $\Omega_{loc}^{q,n-k}(G, \infty)$ it is enough to show that for every ball $B_R = B_R(0,R) \subset \mathbb{R}^{n-1}$ and $T \in \mathbb{R}$ the $(n-k)$-form $\beta$ is $q$-integrable on $Z = B_R \times (-\infty, T)$. Using Lemma 3.4.4 we have

$$\begin{aligned}
\|\beta\|_{L^q,Z}^q &= \int_Z |dx_k \wedge \cdots \wedge dx_{n-1}|_{(x,t)}^q dV(x,t) \\
&\preceq \int_{-\infty}^{T} \int_{B_R} e^{qt(\lambda_k+\cdots+\lambda_{n-1})} e^{-t\mathrm{tr}(\alpha)} dx dt \\
&= \mathrm{Vol}(B_R) \int_{-\infty}^{T} e^{t(q(\lambda_k+\cdots+\lambda_{n-1})-\mathrm{tr}(\alpha))} dt.
\end{aligned}$$

This last integral converges if, and only if, $q > \frac{\mathrm{tr}(\alpha)}{\lambda_k+\cdots+\lambda_{n-1}}$, that is equivalent to $p < \frac{\mathrm{tr}(\alpha)}{w_{k-1}}$.

We now prove (a): Let $B_{R_1} \in \mathbb{R}^{n-1}$ be a ball such that $\mathrm{supp}(f) \subset B_{R_1}$. For $t < 1$ consider $Z_t = B_{R_1} \times [t, 1]$. Since $|\omega \wedge \beta|$ is in $L^1(G)$ because of Hölder's inequality, we have

$$\begin{aligned}\int_G \omega\wedge\beta &= \lim_{t\to-\infty}\int_{Z_t} d(fg\ dx_1\wedge\cdots\wedge dx_{n-1})\\ &= \lim_{t\to-\infty}\int_{B_{R_1}\times\{t\}} fg\ dx_1\wedge\cdots\wedge dx_{n-1}\\ &= \int_{B_{R_1}} f\ dx_1\wedge\cdots\wedge dx_{n-1}\neq 0.\end{aligned}$$

In the second equality we use Stokes theorem.

In order to prove (b) we take $\vartheta\in L^p\Omega^{k-1}(G,\infty)$. There exist two constant $R_2, T_2 > 0$ such that the support of $\vartheta$ is contained in $B_{R_2}\times(-\infty, T_2]$. By Stokes theorem

$$\nu_\beta(d\vartheta)=\int_G d\vartheta\wedge\beta=\lim_{t\to-\infty}\int_{B_{R_2}\times[t,T_2]} d\vartheta\wedge\beta=\lim_{t\to-\infty}\int_{B_{R_2}\times\{t\}}\vartheta\wedge\beta.$$

In the second equality we use again that $|d\vartheta\wedge\beta|$ is in $L^1(G)$. Suppose that $\nu_\beta(d\vartheta)\neq 0$, then there exist $\epsilon>0$ and $t_0$ such that for all $t\leq t_0$,

$$\left|\int_{B_{R_2}\times\{t\}}\vartheta\wedge\beta\right|>\epsilon. \tag{3.9}$$

Assume that

$$\vartheta=\sum_{1\leq i_1<\ldots<i_{k-1}\leq n} a_{i_1,\ldots,i_{k-1}}dx_{i_1}\wedge\ldots\wedge dx_{i_{k-1}}.$$

Therefore

$$\int_{B_{R_2}\times\{t\}}\vartheta\wedge\beta=\int_{B_{R_2}\times\{t\}} a_{1,\ldots,k-1}dx_1\wedge\ldots\wedge dx_{n-1}. \tag{3.10}$$

To simplify the notation we write $a=a_{1,\ldots,k-1}$. Observe that $|\vartheta|_{(x,t)}\geq |a\ dx_1\wedge\ldots\wedge dx_{k-1}|_{(x,t)}$, then

$$\begin{aligned}\|\vartheta\|^p_{L^p} &\geq \int_G |a\ dx_1\wedge\ldots\wedge dx_{k-1}|^p_{(x,t)}dV(x,t)\\ &= \int_{-\infty}^{T_2}\int_{B_{R_2}} |a\ dx_1\wedge\ldots\wedge dx_{k-1}|^p_{(x,t)}e^{-t\mathrm{tr}(\alpha)}dxdt\\ &\succeq \int_{-\infty}^{t_0}\left(\int_{B_{R_2}}|a(x,t)|^p dx\right)e^{t(pw_{k-1}-\mathrm{tr}(\alpha))}dt\\ &\succeq \epsilon\int_{-\infty}^{t_0} e^{t(pw_{k-1}-\mathrm{tr}(\alpha))}dt=+\infty.\end{aligned}$$

In the last line we use (3.9), (3.10) and Jensen's inequality. Since $\vartheta$ is in $L^p\Omega^{k-1}(G,\infty)$ we conclude that (3.9) must be false and as a consequence $\nu_\beta(d\vartheta)=0$. $\square$

Finally, we prove the last part of Theorem 3.4.1 in the diagonal case:

**Proposition 3.4.8.** *If* $p = \frac{\mathrm{tr}(\alpha)}{w_{k-1}}$, *then* $L^pH^k(G,\infty) \neq 0$.

*Proof.* We consider $\omega$ and $\beta$ as in the proof of Proposition 3.4.7. The main difficulty to apply the previous argument in this case is that $\beta$ does not belong to $\Omega^{q,n-k}_{\mathrm{loc}}(G,\infty)$, then $\nu_\beta$ is not well-defined. An alternative is to consider the function

$$\tilde{\nu}_\beta : L^p\Omega^k(G,\infty) \to [0,+\infty],\ \tilde{\nu}_\beta(\varpi) = \liminf_{t\to-\infty} \left| \int_{\mathbb{R}^{n-1}\times[t,+\infty)} \varpi \wedge \beta \right|,$$

which is well-defined because $\mathrm{supp}(\varpi) \cap (\mathbb{R}^{n-1} \times [t,+\infty))$ is compact for every $t \in \mathbb{R}$.

It is clear that

$$\tilde{\nu}_\beta(\omega) = \int_{\mathbb{R}^{n-1}} f(x)\ dx \neq 0.$$

Furthermore we can show using the above argument that $\tilde{\nu}_\beta(d\vartheta) = 0$ for all $\vartheta \in L^p\Omega^{k-1}(G,\infty)$. This implies that $\omega$ represents a non-zero class in $L^p$-cohomology relative to $\infty$. $\square$

### 3.4.2 Non-diagonalizable case

We rename the eigenvalues of $\alpha$ by $\mu_1 < \cdots < \mu_d$, with $d \in \{1,\dots,n-1\}$. Fix a Jordan basis of $\mathbb{R}^{n-1}$,

$$\mathcal{B} = \{e^\ell_{ij} : i = 1,\dots,d; j = 1,\dots,r_i; \ell = 1,\dots,m_{ij}\},$$

where $r_i$ is the dimension of the $\mu_i$-eigenspace spanned by $\{e^1_{i1},\dots,e^1_{ir_i}\}$, $m_{ij}$ is the size of the $j$-Jordan subblock associated to $\mu_i$, and $\alpha(e^\ell_{ij}) = \mu_i e^\ell_{ij} + e^{\ell-1}_{ij}$ for all $\ell = 2,\dots,m_{ij}$. We can write

$$\mathbb{R}^{n-1} = \bigoplus_{i,j} V_{ij},\ \text{where } V_{ij} = \mathrm{Span}(\{e^\ell_{ij} : \ell = 1,\dots,m_{ij}\}). \tag{3.11}$$

Let us denote by $\frac{\partial}{\partial t}$ the unit positive vector which span the factor $\mathbb{R}$ of $G$ and by $dt$ the 1-form associated to $\frac{\partial}{\partial t}$. The 1-forms associated to the dual basis of $\mathcal{B}$ are denoted by $dx^\ell_{ij}$. We put on $G$ the left-invariant Riemannian metric that makes the basis $\mathcal{B} \cup \{\frac{\partial}{\partial t}\}$ orthonormal in $T_eG$.

Observe that

$$e^{t\alpha}e^\ell_{ij} = e^{t\mu_i}\left(e^\ell_{ij} + te^{\ell-1}_{ij} + \ldots + \frac{t^{\ell-1}}{(\ell-1)!}e^1_{ij}\right).$$

This implies

$$L^*_{(x,t)}dx^\ell_{ij} = e^{t\mu_i}\left(dx^\ell_{ij} + \ldots + \frac{t^{m_{ij}-\ell}}{(m_{ij}-\ell)!}dx^{m_{ij}}_{ij}\right).$$

For every $k = 1, \dots, n-1$ we denote by $\Delta_k$ the set of multi-indices

$$I = (i_1, \dots, i_k, j_1, \dots, j_k, \ell_1, \dots, \ell_k) \tag{3.12}$$

with $i_h = 1, \dots, d$, $j_h = 1, \dots, r_{i_h}$ and $\ell_h = 1, \dots, m_{i_h j_h}$ for every $h = 1, \dots, k$. We assume also that the function $h \mapsto (i_h, j_h, \ell_h)$ is injective and preserves the lexicographic order. For a multi-index as (3.12) we write

$$dx_I = dx^{\ell_1}_{i_1 j_1} \wedge \dots \wedge dx^{\ell_k}_{i_k j_k}, \text{ and } w_I = \mu_{i_1} + \dots + \mu_{i_k}.$$

Consider in $\Delta_1$ the lexicographic order and $\zeta : \Delta_1 \to \{1, \dots, n-1\}$ the order-preserving bijection. We denote $dx_h = dx^\ell_{ij}$ if $h = \zeta(i, j, \ell)$.

We have the following general version of Lemma 3.4.4:

**Lemma 3.4.9.** *(i) For every $I \in \Delta_k$ there exists a positive polynomial $P_I$ such that*

$$|dx_I|_{(x,t)} \asymp e^{tw_I} \sqrt{P_I(t)}.$$

*(ii) The volume form on $G$ is $dV(x,t) = e^{-t\mathrm{tr}(\alpha)} dx_1 \wedge \dots \wedge dx_{n-1} \wedge dt$.*

We say that a polynomial $P$ is *positive* if $P(t) > 0$ for all $t \in \mathbb{R}$. Observe that the class of positive polynomials is closed under the sum and the product.

*Proof.* (i) As in the diagonalizable case we consider the left-invariant inner product $\langle\langle\ ,\ \rangle\rangle_{(x,t)}$ on $\Lambda^k(T_{(x,t)}G)$ such that the basis $\{dx_I : I \in \Delta_k\}$ is orthonormal in $\Lambda(T_0 G)$. The induced norm is again denoted by $[\ \ ]_{(x,t)}$. Then

$$\begin{aligned}
[dx^{\ell_1}_{i_1 j_1} \wedge \dots \wedge dx^{\ell_k}_{i_k j_k}]^2_{(x,t)} &= [(L^*_{(x,t)} dx^{\ell_1}_{i_1 j_1}) \wedge \dots \wedge (L^*_{(x,t)} dx^{\ell_k}_{i_k j_k})]^2_0 \\
&= e^{2t(\mu_{i_1} + \dots + \mu_{i_k})} \left[ \left( dx^{\ell_1}_{i_1 j_1} + \dots + \frac{t^{m_{i_1 j_1} - \ell_1}}{(m_{i_1 j_1} - \ell_1)!} dx^{m_{i_1 j_1}}_{i_1 j_1} \right) \wedge \right. \\
&\left. \dots \wedge \left( dx^{\ell_k}_{i_k j_k} + \dots + \frac{t^{m_{i_k j_k} - \ell_k}}{(m_{i_k j_k} - \ell_k)!} dx^{m_{i_k j_k}}_{i_k j_k} \right) \right]^2_0
\end{aligned}$$

From this expression it is easy to extract the polynomial $P_I$. Then the equivalence between $[\ \ ]_{(x,t)}$ and $|\ \ |_{(x,t)}$ implies $(i)$.

(ii) As in Lemma 3.4.4 it is enough to prove that $dV(x,t)(v_1, \dots, v_n) = 1$ for some positive orthonormal basis $\{v_1, \dots, v_n\} \subset T_{(x,t)}G$. Since $\mathcal{B} \cup \{\frac{\partial}{\partial t}\}$ is orthonormal in $T_0 G$, the basis

$$\begin{aligned}
\mathcal{B}_t \cup \{\tfrac{\partial}{\partial t}\} &= \{d_0 L_{(x,t)}(e^\ell_{ij}) : i = 1, \dots, d; j = 1, \dots, r_i; l = 1, \dots, m_{ij}\} \cup \{\tfrac{\partial}{\partial t}\} \\
&= \left\{ e^{t\lambda_i} \left( e^\ell_{ij} + \dots + \frac{t^{\ell-1}}{(\ell-1)!} e^1_{ij} \right) : i = 1, \dots, d; j = 1, \dots, r_i; \ell = 1, \dots, m_{ij} \right\} \cup \{\tfrac{\partial}{\partial t}\}
\end{aligned}$$

is orthonormal in $T_{(x,t)}G$. Then we can check the equality evaluating $dV(x,t)$ in the elements of $\mathcal{B}_t \cup \{\frac{\partial}{\partial t}\}$.

□

We need to estimate the contraction of the vertical flow $\varphi_t$ in this case. To this end we define another left-invariant norm on $G$: For every $v \in \mathbb{R}^n$ we write

$$v = \sum_{i,j} v_{ij} + a\tfrac{\partial}{\partial t}, \tag{3.13}$$

where the first sum corresponds to decomposition (3.11). Given a point $(x,t) \in G$ we define

$$\langle v \rangle_{(x,t)} = \sum_{i,j} \|v_{ij}\|_{(x,t)} + |a|.$$

Using that subspaces $V_{ij}$ are invariant by $e^{t\alpha}$ we can easily see that the norm $\langle\ \rangle_{(x,t)}$ is left-invariant and as a consequence equivalent to the Riemannian norm $\|\ \|_{(x,t)}$. This gives us the following lemma:

**Lemma 3.4.10.** *Let $\omega$ be a $k$-form on $G$, then*

$$|\omega|_{(x,t)} \asymp \sup\{|\omega_{(x,t)}(v_1, \ldots, v_k)| : \langle v_i \rangle_{(x,t)} = 1 \textit{ for all } i = 1, \ldots, k\},$$

*with constant independent of $\omega$ and the point $(x,t) \in G$.*

A set of vectors in $\mathbb{R}^{n-1}$ is said to be $\alpha$-*linearly independent* (denoted also $\alpha$-LI) if it can be extended to a basis of the form $\bigcup_{i,j} \mathcal{B}_{ij}$, where $\mathcal{B}_{ij}$ is a basis of $V_{ij}$.

**Lemma 3.4.11.** *If $\omega$ is a horizontal $k$-form, then the supremum in Lemma* 3.4.10 *is reached on an $\alpha$-LI set.*

Observe that in the previous lemma, since $\omega$ is horizontal, we can think of $\omega_{(x,t)}$ as an alternating $k$-linear map on $\mathbb{R}^{n-1}$.

*Proof.* Since the closed ball for the norm $\langle\ \rangle_{(x,t)}$ is compact, the supremum is reached on a set of vectors $v_1, \ldots, v_k \in \mathbb{R}^{n-1}$, with $\langle v_\ell \rangle_{(x,t)} = 1$ for all $\ell = 1, \ldots, k$. We write these vectors as in (3.13):

$$v_\ell = \sum (v_\ell)_{ij}.$$

Then

$$\begin{aligned}
|\omega_{(x,t)}(v_1, \ldots, v_k)| &= \left|\sum_{i,j} \omega_{(x,t)}((v_1)_{ij}, v_2, \ldots, v_k)\right| \\
&\leq \sum_{i,j} \|(v_1)_{ij}\|_{(x,t)} \left|\omega_{(x,t)}\left(\frac{(v_1)_{ij}}{\|(v_1)_{ij}\|_{(x,t)}}, v_2, \ldots, v_k\right)\right|.
\end{aligned}$$

Since $\langle v_1 \rangle_{(x,t)} = \sum_{i,j} \|(v_1)_{ij}\|_{(x,t)} = 1$, there exists a pair $(i_1, j_1)$ such that

$$|\omega_{(x,t)}(v_1, \ldots, v_k)| \leq \left|\omega_{(x,t)}\left(\frac{(v_1)_{i_1j_1}}{\|(v_1)_{i_1j_1}\|_{(x,t)}}, v_2, \ldots, v_k\right)\right|. \tag{3.14}$$

Observe that the vector $u_1 = \frac{(v_1)_{i_1j_1}}{\|(v_1)_{i_1j_1}\|_{(x,t)}}$ is unitary with respect to the norm $\langle\ \rangle_{(x,t)}$ and it is in $V_{i_1j_1}$. This implies that the inequality (3.14) is in fact an equality. Continuing in this way we can construct an $\alpha$-LI set $\{u_1,\ldots,u_k\}$ that satisfies what we wanted. $\square$

**Lemma 3.4.12.** *If $v \in V_{ij}$, there exists a positive polynomial $P_{ij}$ such that for all $(x,s) \in G$ and $t \geq 0$ we have*

$$\|v\|_{(x,s+t)} \leq e^{-t\mu_i}\sqrt{P_{ij}(t)}\|v\|_{(x,s)}.$$

*Proof.* Observe that for every $s \in \mathbb{R}$ we have

$$\|v\|_{(x,s)} = \|e^{-s\alpha}v\|_0 = e^{-s\mu_i}\|e^{-sJ}v\|_0,$$

where $J$ is the $(m_{ij} \times m_{ij})$-matrix

$$J = J(m_{ij}) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

Then

$$\|v\|_{(x,s+t)} = e^{-(s+t)\mu_i}\|e^{-tJ}(e^{-sJ}v)\|_0 \leq e^{-(s+t)\mu_i}|e^{-tJ}|\|e^{-sJ}v\|_0 = e^{-t\mu_i}|e^{-tJ}|\|v\|_{(x,s)}.$$

Here $|e^{-tJ}|$ denotes the operator norm of the matrix $e^{-tJ}$. Since all norms on $\mathbb{R}^{m_{ij}^2}$ are Lipschitz equivalent, there exists a constant $C_{ij} > 0$, depending only on $m_{ij}$, such that

$$|e^{-tJ}| \leq C_{ij}\sqrt{\sum_{1\leq \ell,r\leq m_{ij}} a_{\ell,r}(t)^2},$$

where $a_{\ell,r}$ are the entries of $e^{-tJ}$. Notice that they are polynomials in $t$, in particular $a_{\ell,\ell} = 1$ for every $\ell = 1,\ldots,m_{ij}$, then the Lemma follows taking

$$P_{ij}(t) = C_{ij}^2 \sum_{1\leq \ell,r\leq m_{ij}} a_{\ell,r}(t)^2.$$

$\square$

Now we are ready to prove the general version of Lemma 3.4.5.

**Lemma 3.4.13.** *If $\omega$ is a horizontal $k$-form on $G$, then there exists a positive polynomial $Q$ such that*

$$|\varphi_t^*\omega|_{(x,s)} \preceq e^{-tw_k}\sqrt{Q(t)}|\omega|_{(x,s+t)}\ \forall t \geq 0.$$

*Proof.* Using Lemmas 3.4.10 and 3.4.11 we have

$$|\varphi_t^*\omega|_{(x,t)} \asymp \max\left\{\left|\varphi_t^*\omega_{(x,s)}\left(\frac{v_1}{\|v_1\|_{(x,s)}},\ldots,\frac{v_k}{\|v_k\|_{(x,s)}}\right)\right| : \{v_1,\ldots,v_k\} \text{ is } \alpha-\mathrm{LI}\right\}$$
$$= \max\left\{\prod_{\ell=1}^{k}\frac{\|v_\ell\|_{(x,s+t)}}{\|v_\ell\|_{(x,s)}}\left|\omega_{(x,s+t)}\left(\frac{v_1}{\|v_1\|_{(x,s+t)}},\ldots,\frac{v_k}{\|v_k\|_{(x,s+t)}}\right)\right| : \{v_1,\ldots,v_k\} \text{ is } \alpha-\mathrm{LI}\right\}$$

Suppose that $v_\ell \in V_{i_\ell j_\ell}$ for every $\ell = 1,\ldots,k$, then by Lemma 3.4.12 and the fact that we are considering $\alpha$-LI sets we obtain

$$|\varphi_t^*\omega|_{(x,t)} \preceq e^{-tw_k}\sqrt{Q(t)}|\omega|_{(x,s+t)},$$

where $Q = \prod_{ij} P_{ij}$. □

Using Lemmas 3.4.9 and 3.4.13 we can easily adapt Proposition 3.4.6 to the general case. The generalization of Proposition 3.4.7 is a bit more complicated.

*Proof of Proposition 3.4.7 in the general case.* We consider again the closed forms

$$\omega_{(x,t)} = d(f(x)g(t)\ dx_1 \wedge \cdots \wedge dx_{k-1}) \text{ and } \beta = dx_k \wedge \cdots \wedge dx_{n-1}.$$

By Lemma 3.4.9 there exists a positive polynomial $P$ such that

$$\|\omega\|_{L^p}^p \preceq \|fg\ dt \wedge dx_1 \wedge \cdots \wedge dx_{k-1}\|_{L^p}^p + \int_{-\infty}^{2} e^{t(pw_k - \mathrm{tr}(\alpha))}P(t)^{\frac{p}{2}}dt.$$

Then $\omega \in L^p\Omega^k(G,\infty)$ for all $p > \frac{\mathrm{tr}(\alpha)}{w_k}$. In a similar way as in the diagonal case we can show that $\beta$ is in $\Omega_{loc}^{q,n-k}(G,\infty)$ if $q > \frac{\mathrm{tr}(\alpha)}{\lambda_k+\cdots+\lambda_{n-1}}$, which is equivalent to $p < \frac{\mathrm{tr}(\alpha)}{w_{k-1}}$. It is also clear that $\nu_\beta(\omega) \neq 0$.

Let us take $\vartheta \in L_\infty^p\Omega^{k-1}(G)$ and prove that $\nu_\beta(d\vartheta) = 0$. Here we find a problem to reproduce the previous argument: It is not clear that $|\vartheta|_{(x,t)} \geq |a_I dx_I|_{(x,t)}$, where $\vartheta = \sum a_I dx_I$, because the Jordan basis is not orthogonal in all tangent spaces. A way to solve it is to consider the forms

$$(\tilde{\upsilon}_I)_{(x,t)} = (L_{(x,t)}^{-1})^* dx_I.$$

If $I = (i_1,\ldots,i_{k-1},j_1,\ldots,j_{k-1},\ell_1,\ldots,\ell_{k-1})$ we have

$$(\tilde{\upsilon}_I)_{(x,t)} = (L_{(x,t)}^{-1})^*(dx_{i_1j_1}^{\ell_1} \wedge \cdots \wedge dx_{i_{k-1}j_{k-1}}^{\ell_{k-1}}) = (L_{(x,t)}^{-1})^* dx_{i_1j_1}^{\ell_1} \wedge \cdots \wedge (L_{(x,t)}^{-1})^* dx_{i_{k-1}j_{k-1}}^{\ell_{k-1}}$$
$$= e^{-tw_I}\left(\sum_{h=0}^{M_1}\frac{(-t)^h}{h!}dx_{i_1j_1}^{\ell_i+h}\right) \wedge \cdots \wedge \left(\sum_{h=0}^{M_{k-1}}\frac{(-t)^h}{h!}dx_{i_{k-1}j_{k-1}}^{\ell_{k-1}+h}\right),$$

where $M_s = m_{i_sj_s} - \ell_s$. We define $(\upsilon_I)_{(x,t)} = e^{tw_I}(\tilde{\upsilon}_I)_{(x,t)}$ and write

$$\vartheta = \sum_{I \in \Delta_{k-1}} a_I \upsilon_I.$$

Observe that $|\upsilon_I|_{(x,t)} \asymp e^{tw_I}$ for every $(x,t) \in G$.

Since $\{\upsilon_I : I \in \Delta_{k-1}\}$ is orthogonal at every point with respect to $\langle\langle\ ,\ \rangle\rangle_{(x,t)}$, then $[\vartheta]_{(x,t)} \geq [a_I \upsilon_I]_{(x,t)}$ for all $I \in \Delta_{k-1}$ and therefore $|\vartheta|_{(x,t)} \succeq |a_I \upsilon_I|_{(x,t)}$.

We can easily observe that

$$\vartheta \wedge \beta = a_{I_0} dx_1 \wedge \cdots \wedge dx_{n-1},$$

where $I_0$ is such that $dx_{I_0} = dx_1 \wedge \ldots \wedge dx_{k-1}$. Suppose that $\nu_\beta(d\vartheta) \neq 0$, then if $\mathrm{supp}(\vartheta) \subset B_R \times (-\infty, T]$ with $B_R = B(0,R) \subset \mathbb{R}^{n-1}$, there exist $\epsilon > 0$ and $t_0$ such that for all $t \leq t_0$,

$$\left| \int_{B_R \times \{t\}} a_{I_0}(x,t) dx \right| > \epsilon.$$

Now we have

$$\begin{aligned}
\|\vartheta\|_{L^p}^p &\succeq \int_G |a_{I_0} \upsilon_{I_0}|_{(x,t)}^p dV(x,t) \\
&\succeq \int_{-\infty}^{t_0} \left( \int_{B_R} |a_{I_0}(x,t)|^p dx \right) e^{t(pw_{k-1} - \mathrm{tr}(\alpha))} dt \\
&\succeq \epsilon^p \int_{-\infty}^{t_0} e^{t(pw_{k-1} - \mathrm{tr}(\alpha))} dt = +\infty
\end{aligned}$$

This contradiction proves that $\nu_\beta(d\gamma) = 0$. $\square$

Using $\tilde{\nu}_\beta$ as in Proposition 3.4.8 and the above argument it is easy to prove that $L^p H^k(G, \xi) \neq 0$ for $p = \frac{\mathrm{tr}(\alpha)}{w_{k-1}}$, which finishes the proof of Theorem 3.4.1 in the general case.

# Chapter 4

# Relative Orlicz cohomology

The aim of this chapter is to prove Theorem 1.2.8. Before that we prove the quasi-isometry invariance of the simplicial version of relative Orlicz cohomology.

## 4.1 Simplicial relative Orlicz cohomology and quasi-isometry invariance

Let $X$ be a finite-dimensional simplicial complex with bounded geometry and $\phi$ a Young function. As in the $L^p$-case we have the following proposition:

**Proposition 4.1.1.** *The usual coboundary operator $\delta = \delta_k : \ell^\phi C^k(X) \to \ell^\phi C^{k+1}(X)$ is continuous.*

*Proof.* Let $\theta$ be a cochain in $\ell^\phi C^k(X)$, then

$$\begin{aligned}\|\delta\theta\|_{L^\phi} &= \inf\left\{\gamma > 0 : \sum_{\sigma\in X_{k+1}} \phi\left(\frac{\delta\theta(\sigma)}{\gamma}\right) \le 1\right\}\\ &= \inf\left\{\gamma > 0 : \sum_{\sigma\in X_{k+1}} \phi\left(\frac{\theta(\partial\sigma)}{\gamma}\right) \le 1\right\}.\end{aligned}$$

The bounded geometry implies that there is a constant $N(1)$ such that every $k$-simplex $\tau$ in $X$ is on the boundary of at most $N(1)$ $k$-simplices. Then

$$\sum_{\sigma\in X_{k+1}} \phi\left(\frac{\theta(\partial\sigma)}{\gamma}\right) \le \sum_{\tau\in X_k} N(1)\phi\left(\frac{\theta(\tau)}{\gamma}\right),$$

which implies

$$\|\delta\theta\|_{L^\phi} \le \inf\left\{\gamma > 0 : \sum_{\tau\in X_k} N(1)\phi\left(\frac{\theta(\tau)}{\gamma}\right) \le 1\right\} = \|\theta\|_{L^{N(1)\phi}}.$$

The proof ends using the equivalence between $\| \ \|_{L^{N(1)\phi}}$ and $\| \ \|_{L^\phi}$ (Remark 2.5.1). $\square$

The quasi-isometry invariance of the simplicial relative Orlicz cohomology is proved in a similar way as the $L^p$-case. The only difference is the continuity of the pull-back and the homotopy defined in the proof of Theorem 1.2.1.

*Proof of Theorem 1.2.8.* Consider the family of maps $c_F : C_k(X) \to C_k(Y)$ given in Lemma 3.1.2. We consider the pull-back of a $k$-cochain $\theta \in \ell^\phi C^k(Y, F(\xi))$ as

$$F^*\theta = \theta \circ c_F.$$

It is clear that $\delta \circ F = F \circ \delta$ and that $F^*\theta$ vanishes on a neighborhood of $\xi$. Let us prove that $F^*$ is well-defined and continuous from $\ell^\phi C^k(Y, F(\xi))$ to $\ell^\phi C^k(X, \xi)$:

$$\begin{aligned}
\|F^*\theta\|_{L^\phi} &= \inf\left\{\gamma > 0 : \sum_{\sigma\in X^k} \phi\left(\frac{F^*\theta(\sigma)}{\gamma}\right) \leq 1\right\} \\
&\leq \inf\left\{\gamma > 0 : \sum_{\sigma\in X^k} \phi\left(\frac{N_k}{\gamma}\sum_{\tau\in|c_F(\theta)|} |\theta(\tau)|\right) \leq 1\right\} \\
&\leq \inf\left\{\gamma > 0 : \sum_{\sigma\in X^k}\sum_{\tau\in|c_F(\theta)|} \frac{1}{\ell(c_F(\sigma))}\phi\left(\frac{N_kL_k}{\gamma}|\theta(\tau)|\right) \leq 1\right\},
\end{aligned}$$

where $N_k$ and $L_k$ are constants such that for all $\sigma \in X^k$,

$$\|c_F(\sigma)\|_\infty \leq N_k \text{ and } \ell(c_F(\sigma)) \leq L_k.$$

As in the proof of Theorem 1.2.1 there exists a constant $C$ such that for every $\tau \in Y^k$, then $\tau \in |c_F(\sigma)|$ for at most $C$ simplices $\sigma \in X^k$. Then

$$\|F^*\theta\|_{L^\phi} \leq \inf\left\{\gamma > 0 : \sum_{\sigma\in Y^k} C\phi\left(\frac{N_kL_k}{\gamma}|\theta(\tau)|\right) \leq 1\right\} = N_kL_k\|\theta\|_{L^{C\phi}} \preceq \|\theta\|_{L^\phi}.$$

Therefore $F^*\theta \in \ell^\phi_\xi C^k(X)$ and $F^*$ is continuous.

If $G : X \to Y$ is another quasi-isometry at bounded uniform distance of $F$, Lemma 3.1.3 gives us an homotopy $h$ between $c_F$ and $c_G$, then we consider

$$H : \ell^\phi C^k(Y, F(\xi)) \to \ell^\phi C^{k-1}(X, \xi),\ H\theta(\sigma) = \theta(h(\sigma)).$$

The continuity of $H$ can be proved in a similar way as we have proved that $F^*$ is continuous. It is clear that if $\theta \in \ell^\phi C^k(Y, F(\xi))$, then $H\theta$ vanishes on a neighborhood of $\xi$ because the Hausdorff distance between $c_F(\sigma)$ and $h(\sigma)$ is uniformly bounded.

Observe that $H$ is an homotopy between $F$ and $G$, then they induce the same map in relative $\ell^\phi$-cohomology. As in the proof of Theorem 1.2.1 this implies that if $\overline{F} : Y \to X$ is a quasi-inverse of $F$, then $F^* \circ \overline{F}^*$ and $\overline{F}^* \circ F^*$ are the identity in relative $\ell^\phi$-cohomology, which finishes the proof. $\square$

## 4.2 Equivalence between the two versions in the case of Lie groups

Consider now a Lie group $G$ equipped with a left-invariant Riemannian metric. Given $x \in G$ we denote by $L_x$ and $R_x$ the left and right translation by $x$ respectively, and by $dx$ the Riemannian volume on $G$. The unity of $G$ will be always denoted by $e$. Observe that such a manifold has always bounded geometry.

The aim of this section is to prove Theorem 1.2.9. We assume here that the Young function $\phi$ is doubling.

### 4.2.1 Convolution of locally integrable forms

Let $\kappa : G \to [0,1]$ be a kernel on $G$, which means:

- $\kappa \in C^\infty(G)$,
- $\mathrm{supp}(\kappa)$ is a compact neighborhood of $e \in G$, and
- $\int_G \kappa(x)dx = 1$.

If $\omega$ is a locally integrable $k$-form on $G$ we consider its convolution with $\kappa$ as the $k$-form

$$(\omega * \kappa)_x = \int_G (R_z^*\omega)_x \kappa(z)dz.$$

**Lemma 4.2.1.** *There exists a constant $C > 0$ such that for every locally integrable $k$-form $\omega$ on $G$ and $x \in G$ we have*

$$|\omega * \kappa|_x \leq C|\omega| * \kappa(x),$$

*where $|\omega| * \kappa$ is the convolution of the function $x \to |\omega|_x$ with the kernel $\kappa$.*

*Proof.* Let $v_1, \ldots, v_k$ be vectors in $T_xG$, then

$$\begin{aligned}
|(\omega * \kappa)_x(v_1, \ldots, v_k)| &= \left| \int_G (R_z^*\omega)_x(v_1, \ldots, v_k)\kappa(z)dz \right| \\
&\leq \int_G |(R_z^*\omega)_x(v_1, \ldots, v_k)|\kappa(z)dz \\
&= \int_G |\omega_{xz}(d_xR_z(v_1), \ldots, d_xR_z(v_k))|\kappa(z)dz.
\end{aligned}$$

Since $R_z \circ L_x = L_x \circ R_z$, we have $|d_e(R_z \circ L_x)| = |d_e(L_x \circ R_z)|$ (here $|\ \ |$ is the usual operator norm) and therefore $|d_xR_z \circ d_eL_x| = |d_zL_x \circ d_eR_z|$. Using that $L_x$ is an isometry we obtain $|d_x R_z| = |d_e R_z|$. The function $z \mapsto |d_e R_z|$ is continuous, then it has a maximum $M$ in $\mathrm{supp}(\kappa)$. If $\|v_1\| = \ldots = \|v_k\| = 1$,

$$|\omega_{xz}(d_x R_z(v_1), \ldots, d_x R_z(v_k))| \leq M^k |\omega|_{xz},$$

which implies $|\omega * \kappa|_x \leq C|\omega| * \kappa(x)$ with $C = M^k$. □

A consequence of Lemma 4.2.1 is that the convolution of a locally integrable form is also locally integrable.

**Proposition 4.2.2.** *Let $\omega$ be a locally integrable $k$-form on $G$, then:*

*(i) If $\omega$ has weak derivative $d\omega$, then the convolution $\omega * \kappa$ has weak derivative and*

$$d(\omega * \kappa) = d\omega * \kappa.$$

*(ii) The convolution $\omega * \kappa$ is a differential form.*

*Proof.* (i) For every $z$ we have $d(R_z^* \omega) = R_z^* d\omega$ in a weak sense. To see this take $\beta \in \Omega^{n-k-1}(G)$ with compact support, then

$$\begin{aligned} \int_G (R_z^* d\omega) \wedge \beta &= \int_G R_z^*(d\omega \wedge R_{z^{-1}}^* \beta) \\ &= \int_G d\omega \wedge R_{z^{-1}}^* \beta \\ &= (-1)^{k+1} \int_G \omega \wedge dR_{z^{-1}}^* \beta \\ &= (-1)^{k+1} \int_G (R_z^* \omega) \wedge d\beta. \end{aligned}$$

Therefore the weak derivative with respect to $x \in G$ of the $k$-form $\Phi(x, z) = (R_z^* d\omega)_x \kappa(z)$ is

$$d\Phi(x, z) = (R_z^* d\omega)_x \kappa(z).$$

Since $z \mapsto d\Phi(x, z)$ has compact support for all $x \in G$, by Lemma 2.3.1 we conclude that

$$(d\omega * \kappa) = \int_G (R_z^* d\omega) \kappa(z) dz$$

is the weak derivative of the convolution $\omega * \kappa$.

(ii) Suppose first that $\omega = f$ is a 0-form, which is equivalent to say that it is a locally integrable function on $G$. Consider $Y$ a vector field on $G$ with flow $\varphi_t$. First observe that

$$f * \kappa(x) = \int_G f(xz) \kappa(z) dz = \int_G f(y) \kappa(x^{-1} y) dy.$$

Then

$$L_Y(f*\kappa)(x) = \frac{\partial}{\partial t}\bigg|_{t=0} (f*\kappa(\varphi_t(x)))$$
$$= \frac{\partial}{\partial t}\bigg|_{t=0} \int_G f(y)\kappa(\varphi_t(x)^{-1}y)dy.$$

Since $\varphi$ is smooth and $\kappa$ is smooth with compact support, the classical Leibniz integral Rule implies that this derivative exists and

$$L_X(f*\kappa)(x) = \int_G f(y)\,\frac{\partial}{\partial t}\bigg|_{t=0} \kappa(\varphi_t(x)^{-1}y)dy.$$

Using this arguments we can prove by induction that $L_{Y_m}\dots L_{Y_m}(f*\kappa)(x)$ exists for all $x\in G$ for every family of vector fields $Y_1,\dots,Y_m$. This implies that $f*\kappa$ is smooth.

Now consider $\{e_1,\dots,e_n\}$ a basis of $T_eG$ and $X_1,\dots,X_n$ the right-invariant fields verifying $X_i(e)=e_i$. Let $\varphi_t^i$ be the flow associated to $X_i$ for every $i=1,\dots,n$. If $\omega$ is a $k$-form with $k\geq 1$ we set

$$f_{i_1,\dots,i_k}(x) = (\omega*\kappa)_x(X_{i_1}(x),\dots,X_{i_k}(x)).$$

To prove that $\omega*\kappa$ is smooth it is enough to prove that all these functions are smooth. Observe that if

$$g_{i_1,\dots,i_k}(x) = \omega(x)(X_{i_1}(x),\dots,X_{i_k}(x)),$$

then $f_{i_1,\dots,i_k} = g_{i_1,\dots,i_k}*\kappa$. This reduce the general case to the case $k=0$ and finish the proof.

□

### 4.2.2 Proof of Theorem 1.2.9

Let $\mathcal{U}$ be an open covering in $G$. We consider the following cochain complexes:

- $\mathcal{L}^\phi\Omega^k(G,\mathcal{U})$ is the space of all differential forms $\omega\in\Omega^k(G)$ such that $\omega|_U$ and $d\omega|_U$ are in $L^\phi\Omega^k(U)$ for all $U\in\mathcal{U}$, and the functions $U\mapsto\|\omega|_U\|_{L^\phi}$ and $U\mapsto\|d\omega|_U\|_{L^\phi}$ are in $\ell^\phi(\mathcal{U})$. The norm of $\omega\in\mathcal{L}^\phi\Omega^k(G,\mathcal{U})$ is defined by

$$|\omega|_{\mathcal{L}^\phi} = \|\theta\|_{\ell^\phi} + \|\theta'\|_{\ell^\phi},$$

where $\theta(U)=\|\omega|_U\|_{L^\phi}$ and $\theta'(U)=\|d\omega|_U\|_{L^\phi}$. Naturally, the map defining the chain complex is the usual derivative.

- $\mathcal{I}^\phi\Omega^k(G,\mathcal{U}) = L^\phi\Omega^k(G)\cap\mathcal{L}^\phi\Omega^k(G,\mathcal{U})$ with the norm $|\ |_{\mathcal{I}^\phi} = |\ |_{L^\phi} + |\ |_{\mathcal{L}^\phi}$.

- If $G$ is Gromov-hyperbolic and $\xi$ is a point in $\partial G$, we consider $\mathcal{L}^\phi\Omega^k(G,\mathcal{U},\xi)$ the subcomplex consisting of all forms $\omega \in \mathcal{L}^\phi\Omega^k(G,\mathcal{U})$ that vanish on a neighborhood of $\xi$. In this case we also define $\mathcal{I}^\phi\Omega^k(G,\mathcal{U},\xi) = L^\phi\Omega^k(G,\xi) \cap \mathcal{L}^\phi\Omega^k(G,\mathcal{U},\xi)$.

We consider in this section a uniformly locally finite open covering $\mathcal{U}$ such that all intersections are bi-Lipschitz diffemorphic to the unit ball in the corresponding Euclidean space, and $X_G$ the nerve of $\mathcal{U}$. Remember that a Lie group equipped with a left-invariant metric is always complete and has bounded geometry; also notice that if it is contractible, then it is uniformly contractible. If $G$ is Gromov-hyperbolic and $\xi$ is a point in $\partial G$, then we denote by $\bar{\xi}$ the point in $\partial X_G$ corresponding to $\xi$ by a canonical quasi-isometry.

First of all we see that, unlike the $L^p$-case, $\mathcal{L}^\phi\Omega^k(G,\mathcal{U})$ can be different from $L^\phi\Omega^k(G)$.

**Example 4.2.3.** We take the doubling Young function $\phi : \mathbb{R} \to [0,+\infty)$,

$$\phi(t) = \phi_{p,\kappa}(t) = \frac{|t|^p}{\log(e+|t|^{-1})^\kappa},$$

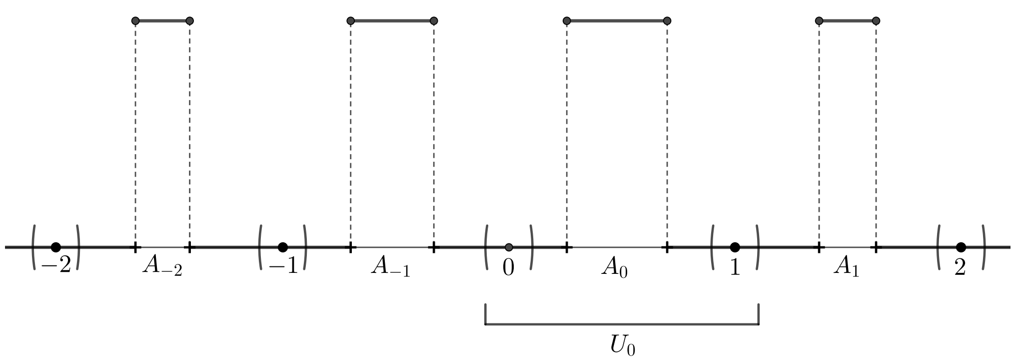
with $p > 1$ and $\kappa > 0$. We want to construct a 1-form $\omega$ in $\mathcal{L}^\phi\Omega^1(\mathbb{R},\mathcal{U})$ and out of $L^\phi\Omega^1(\mathbb{R})$, where $\mathcal{U} = \{U_n = (n-\epsilon, n+1+\epsilon) : n \in \mathbb{Z}\}$ with $\epsilon > 0$ much smaller than 1.

Let $\{a_n\}_{n\in\mathbb{Z}}$ be a sequence of positive numbers such that:

- $\sum a_n^p = +\infty$, and
- $\sum \phi(a_n) < +\infty$.

Take for every $n \in \mathbb{Z}$ an interval $A_n$ in $\mathbb{R}$ such that $A_n \subset (n+2\epsilon, n+1-2\epsilon)$ and $long(A_n) = a_n^p$. We can suppose that $a_n$ is small enough for every $n \in \mathbb{Z}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f = \sum_{n\in\mathbb{Z}} \mathbb{1}_{A_n}.$$



On the one hand

$$\int_{\mathbb{R}} \phi(f(t))dt = \sum_{n\in\mathbb{Z}} \int_n^{n+1} \phi(\mathbb{1}_{A_n}(t))dt = \sum_{n\in\mathbb{Z}} a_n^p \phi(1) = +\infty.$$

But on the other hand

$$\int_{U_n} \phi\left(\frac{f(t)}{\gamma}\right) dt = a_n^p \phi\left(\frac{1}{\gamma}\right) = \frac{a_n^p}{\gamma^p \log(e+\gamma)^\kappa} \leq \left(\frac{a_n}{\gamma}\right)^p,$$

which implies $\|f|_{U_n}\|_{L^\phi} \leq a_n$ and then

$$\sum_{n\in\mathbb{Z}} \phi(\|f|_{U_n}\|_{L^\phi}) \leq \sum_{n\in\mathbb{Z}} \phi(a_n) < +\infty.$$

We can find a smooth function $g$ close enough from $f$ such that $g - f \in L^\phi(\mathbb{R})$ and $\|(g-f)|_{U_n}\|_{L^\phi} \in \ell^\phi(\mathbb{Z})$ and consider the 1-form $\omega = g\ dt$. Since $|\omega|_t = |g(t)|$ and $d\omega = 0$ we can see that $\omega \in \mathcal{L}^\phi\Omega^1(\mathbb{R},\mathcal{U})$ and $\omega \notin L^\phi\Omega^1(\mathbb{R})$.

In this case the other inclusion is true. One can prove that $L^\phi\Omega^k(\mathbb{R}) \subset \mathcal{L}^\phi\Omega^k(\mathbb{R},\mathcal{U})$ for $k = 0, 1$ using the inequality $\phi(s)\phi(t) \leq 2^\kappa\phi(st)$. In fact this inclusion can be proved for every Riemannian manifold with bounded geometry and every doubling Young function satisfying an inequality $\phi(t)\phi(s) \leq C\phi(st)$ for all $s, t \in \mathbb{R}$ and some constant $C$.

Using Lemma 2.5.4 we can easily deduce the following lemma:

**Lemma 4.2.4.** *Consider a sequence $\{\omega_n\}_{n\in\mathbb{N}} \subset \mathcal{L}^\phi\Omega^k(G,\mathcal{U})$ and denote for every $U \in \mathcal{U}$,*

$$\theta_n(U) = \|\omega_n|_U\|_{L^\phi} \text{ and } \theta'_n(U) = \|d\omega_n|_U\|_{L^\phi}.$$

*Then $\omega_n \to 0$ in $\mathcal{L}^\phi\Omega^k(G,\mathcal{U})$ if, and only if,*

$$\sum_{U\in\mathcal{U}} \phi(\theta_n(U)) \to 0 \text{ and } \sum_{U\in\mathcal{U}} \phi(\theta'_n(U)) \to 0. \tag{4.1}$$

**Lemma 4.2.5.** *The space $\Omega_c^k(G)$ of differential $k$-forms with compact support is dense in $\mathcal{L}^\phi\Omega^k(G,\mathcal{U})$.*

*Proof.* Take $\omega \in \mathcal{L}^\phi\Omega^k(G,\mathcal{U})$ and denote $\theta(U) = \|\omega|_U\|_{L^\phi}$ and $\theta'(U) = \|d\omega|_U\|_{L^\phi}$. Since $\phi$ is doubling we have

$$\sum_{U\in\mathcal{U}} \phi(\theta(U)), \sum_{U\in\mathcal{U}} \phi(\theta'(U)) < +\infty.$$

Given $n \in \mathbb{N}$ we take a compact subset $K_n \subset G$ such that

$$\sum_{U\not\subset K_n} \phi(\theta(U)), \sum_{U\not\subset K_n} \phi(\theta'(U)) < \frac{1}{n}.$$

Let $A_n$ be an open set in $G$ such that $K_n \subset A_n$ and $\overline{A_n}$ is compact, and $h_n : G \to [0,1]$ a smooth function with support in $A_n$ such that $h_n|_{K_n} \equiv 1$ and $|dh_n|_x \leq 1$ for all $x \in G$.

Now we define $\omega_n = h_n\omega$ and

$$\theta_n(U) = \|(\omega - \omega_n)|_U\|_{L^\phi} \text{ and } \theta'_n(U) = \|d(\omega - \omega_n)|_U\|_{L^\phi}.$$

We will use Lemma 4.2.4 to prove that $\omega_n \to \omega$ in $\mathcal{L}^\phi\Omega^k(G,\mathcal{U})$.

On the one hand we have $\theta_n(U) = 0$ if $U \subset K_n$ and $|\theta_n(U)| \leq |\theta(U)|$ otherwise. Then

$$\sum_{U\in\mathcal{U}} \phi(\theta_n(U)) \leq \sum_{U\not\subset K_n} \phi(\theta(U)) \to 0, \text{ when } n \to +\infty.$$

On the other hand, $\theta'_n(U) = 0$ if $U \subset K_n$ and $\theta'_n(U) \leq \|dh_n \wedge \omega\|_{L^\phi} + \|(1-h_n)d\omega\|_{L^\phi}$ otherwise. It is easy to see that $|dh_n \wedge \omega|_x \leq |dh_n|_x|\omega|_x \leq |\omega|_x$, then

$$\begin{aligned}
\sum_{U\in\mathcal{U}} \phi(\theta'_n(U)) &\leq \sum_{U\not\subset K_n} \phi(\|\omega|_U\|_{L^\phi} + \|d\omega|_U\|_{L^\phi}) \\
&\leq \sum_{U\not\subset K_n} \frac{\phi(2\|\omega|_U\|_{L^\phi}) + \phi(2\|d\omega|_U\|_{L^\phi})}{2} \\
&\leq \frac{D}{2}\left(\sum_{U\not\subset K_n} \phi(\theta(U)) + \sum_{U\not\subset K_n} \phi(\theta'(U))\right) \to 0,
\end{aligned}$$

when $n \to +\infty$. Here $D$ is the doubling constant of $\phi$. $\square$

The following proposition will be proved in a similar way as Theorem 1.2.3. Some lemmas will be necessary for this purpose.

**Proposition 4.2.6.** *The cochain complexes $(\ell^\phi C^*(X_G), \delta)$ and $(\mathcal{L}^\phi\Omega^*(G,\mathcal{U}), d)$ are homotopically equivalent. So are the cochain complexes $(\ell^\phi C^*(X_G, \bar{\xi}), \delta)$ and $(\mathcal{L}^\phi\Omega^*(G,\mathcal{U},\xi), d)$.*

We need an $L^\phi$-version of Lemma 3.2.1:

**Lemma 4.2.7.** *The cochain complex $(L^\phi\Omega^*(B), d)$, where $B$ is the unit ball in $\mathbb{R}^n$, retracts to the complex $(\mathbb{R} \to 0 \to 0 \to \ldots)$.*

*Proof.* As in the $L^p$-version, for $x \in B$ and $\omega \in L^\phi\Omega^k(B)$ we consider the map

$$\chi_x(\omega) = \int_0^1 \eta_t^*\left(\iota_{\frac{\partial}{\partial t}}\varphi_x^*\omega\right) dt,$$

where $\eta_t : B \to [0,1] \times B$, is defined by $\eta_t(y) = (t,y)$, and $\varphi_x : [0,1] \times B$, $\varphi_x(t,y) = ty + (1-t)x$. If $\omega$ is a $k$-form in $L^\phi\Omega^k(B)$ with $k \geq 1$, we put

$$h(\omega) = \frac{1}{\mathrm{Vol}(\frac{1}{2}B)} \int_{\frac{1}{2}B} \chi_x(\omega)dx,$$

and

$$h(f) = \frac{1}{\mathrm{Vol}(\frac{1}{2}B)} \int_{\frac{1}{2}B} f(x)dx,$$

if $f$ is a function in $L^\phi\Omega^0(B)$. As in Lemma 3.2.1, we have $h \circ d + d \circ h = \mathrm{Id}$. We have to prove that $h$ is $L^\phi$-continuous in all degrees. To this end remember that if $\omega$ is a $k$-form with $k \geq 1$, then

$$|h(\omega)|_y \leq C \int_{B(y,2)} |z-y|^{1-n} u(z)dz,$$

where $u(z) = |\omega|_z$ if $z \in B$ and $u(z) = 0$ if $z \notin B$, and $C$ is a constant. Using this estimate we have

$$\begin{aligned} \|h(\omega)\|_{L^\phi} &= \inf\left\{\gamma > 0 : \int_B \phi\left(\frac{|h(\omega)|_y}{\gamma}\right) dy \leq 1\right\} \\ &\leq \inf\left\{\gamma > 0 : \int_B \phi\left(\int_{B(y,2)} |z-y|^{1-n} \frac{u(z)}{\gamma} dz\right) dy \leq 1\right\}. \end{aligned}$$

Since $\int_{B(y,2)} |z-y|^{1-n} dz < +\infty$, we can use Jensen's inequality and write

$$\begin{aligned} \|h(\omega)\|_{L^\phi} &\leq \inf\left\{\gamma > 0 : \mathrm{Vol}(B(0,3)) \int_B \int_{B(0,3)} \phi\left(\frac{u(z)}{\mathrm{Vol}(B(0,3))\gamma}\right) \frac{dz}{|z-y|^{n-1}} dy \leq 1\right\} \\ &= \inf\left\{\gamma > 0 : \mathrm{Vol}(B(0,2)) \int_{B(0,3)} \phi\left(\frac{u(z)}{\mathrm{Vol}(B(0,3))\gamma}\right) \left(\int_B \frac{dy}{|z-y|^{n-1}}\right) dz \leq 1\right\}. \end{aligned}$$

We have that there exists a constant $K > 0$ such that $\int_B \frac{dy}{|z-y|^{n-1}} \leq K$ for all $z \in B(0,3)$, then

$$\|h(\omega)\|_{L^\phi} \leq \mathrm{Vol}(B(0,2)) \|\omega\|_{L^{\tilde{K}\phi}} \preceq \|\omega\|_{L^\phi},$$

where $\tilde{K} = K\,\mathrm{Vol}(B(0,3))$.

As in Lemma 3.2.1, the identity $dh(\omega) = \omega - h(d\omega)$ and the above estimates give us the continuity of $h$ for the norm $|\ \ |_{L^\phi}$ in all degrees. $\square$

In a similar way as in Lemma 3.2.2, we have:

**Lemma 4.2.8.** *Let $M$ and $N$ be two Riemannian manifolds and $f : M \to N$ a bi-Lipschitz diffeomorphism with constant $L$. Then for all $k \in \mathbb{N}$ the pull-back $f^* : L^\phi\Omega^k(N) \to L^\phi\Omega^k(M)$ is continuous and its operator norm is bounded depending on $L$, $k$, $\phi$ and $n = \dim(M)$.*

*Proof of Proposition 4.2.6.* Let us define the bicomplex

$$C^{k,\ell} = \left\{\omega \in \prod_{U \in \mathcal{U}_\ell} L^\phi\Omega(U) : \{\|\omega_U\|_{L^\phi}\}_{U\in\mathcal{U}_\ell}, \{\|d\omega_U\|_{L^\phi}\}_{U\in\mathcal{U}_\ell} \in \ell^\phi(\mathcal{U}_\ell)\right\},$$

equipped with the norm

$$\|\omega\| = \|\theta\|_{\ell^\phi} + \|\theta'\|_{\ell^\phi},$$

where $\theta(U) = \|\omega_U\|_{L^\phi}$ and $\theta'(U) = \|d\omega_U\|_{L^\phi}$. The derivatives are defined as in the proof of Theorem 1.2.3, by

- $(d'\omega)_U = (-1)^\ell d\omega_U$ for all $\omega \in C^{k,\ell}$,
- If $\omega \in C^{k,\ell}$ and $W \in \mathcal{U}_{\ell+1}$, $W = U_0 \cap \ldots \cap U_{\ell+1}$, then

$$(d''\omega)_W = \sum_{i=0}^{\ell+1} (-1)^i (\omega_{U_0 \cap \ldots U_{i-1} \cap U_{i+1} \cap \ldots \cap U_{\ell+1}})|_W.$$

It is easy to see that $d'$ and $d''$ are well-defined and continuous, and that $d' \circ d'' + d'' \circ d' = 0$.

Observe that, as in the $L^p$-case, $\operatorname{Ker} d'|_{C^{0,\ell}}$ is isomorphic to $\ell^\phi C^\ell(X)$ and $d''$ corresponds with the coboundary operator $\delta$. On the other hand, it is clear that $\operatorname{Ker} d''|_{C^{k,0}} = \mathcal{L}^\phi\Omega(G, \mathcal{U})$.

<u>Claim 1</u>: The cochain complex $(C^{*,\ell}, d')$ retracts to $(\operatorname{Ker} d'|_{C^{0,\ell}} \to 0 \to \ldots)$ for all $\ell \in \mathbb{N}$.

For every $U \in \mathcal{U}_\ell$ consider $f_U : U \to B$ an $L$-bi-Lipschitz diffeomorphism ($L$ does not depend on $U$ and $B$ is the unit ball in the corresponding Euclidean space). We define $H : C^{k,\ell} \to C^{k-1,\ell}$ by

$$(H\omega)_U = (-1)^\ell f_U^* h (f_U^{-1})^* \omega_U,$$

where $h : L^\phi\Omega^k(B) \to L^\phi\Omega^{k-1}(B)$ is the map given by Lemma 4.2.7. Here we are using the identification $C^{-1,\ell} = \operatorname{Ker} d'|_{C^{0,\ell}}$. One can easily verify that $Hd' + d'H = \mathrm{Id}$. Using Lemma 4.2.8 we have that $H$ is continuous.

<u>Claim 2</u>: The cochain complex $(C^{k,*}, d'')$ retracts to $(\operatorname{Ker} d''|_{C^{k,0}} \to 0 \to \ldots)$ for all $k \in \mathbb{N}$.

We define $P : C^{k,\ell} \to C^{k,\ell-1}$ in the same way as in the $L^p$-case. It is easy to prove that $P$ is continuous and $Pd'' + d''P = \mathrm{Id}$. Here $C^{k,-1} = \operatorname{Ker} d''|_{C^{k,0}}$.

Using Lemma 2.2.1 we obtain the equivalence between $\mathcal{L}^\phi\Omega^k(M)$ and $\ell^\phi C^k(X)$.

To prove the relative case we have to consider the bicomplex $(C_\xi^{*,*}, d', d'')$, where $C_\xi^{k,\ell}$ is the subspace consisting of the elements $\omega$ of $C^{k,\ell}$ for which there exists $V \subset \overline{G}$ a neighborhood of $\xi$ such that $\omega_U \equiv 0$ if $U \subset V$. The above argument works in this case because all maps preserve the subspaces $C_\xi^{k,\ell}$. □

**Proposition 4.2.9.** *The cochain complexes $(\mathcal{L}^\phi\Omega^*(G, \mathcal{U}), d)$ and $(\mathcal{I}^\phi\Omega^*(G, \mathcal{U}), d)$ are homotopically equivalent. So are the cochain complexes $(\mathcal{L}^\phi\Omega^*(G, \mathcal{U}, \xi), d)$ and $(\mathcal{I}^\phi\Omega^*(G, \mathcal{U}, \xi), d)$.*

Combining Propositions 4.2.6 and 4.2.9 we have the following diagram:

$$\begin{tikzcd}
\ell^\phi C^0(X_G) \arrow[r, "\delta"] \arrow[d, bend right, "\sim"'] & \ell^\phi C^1(X_G) \arrow[r, "\delta"] \arrow[d, bend right, "\sim"'] & \ell^\phi C^2(X_G) \arrow[r, "\delta"] \arrow[d, bend right, "\sim"'] & \cdots \\
\mathcal{L}^\phi\Omega^0(G,\mathcal{U}) \arrow[r, "d"] \arrow[u, bend right] \arrow[d, bend right, "\sim"'] & \mathcal{L}^\phi\Omega^1(G,\mathcal{U}) \arrow[r, "d"] \arrow[u, bend right] \arrow[d, bend right, "\sim"'] & \mathcal{L}^\phi\Omega^2(G,\mathcal{U}) \arrow[r, "d"] \arrow[u, bend right] \arrow[d, bend right, "\sim"'] & \cdots \\
\mathcal{I}^\phi\Omega^0(G,\mathcal{U}) \arrow[r, "d"] \arrow[u, bend right] & \mathcal{I}^\phi\Omega^1(G,\mathcal{U}) \arrow[r, "d"] \arrow[u, bend right] & \mathcal{I}^\phi\Omega^2(G,\mathcal{U}) \arrow[r, "d"] \arrow[u, bend right] & \cdots
\end{tikzcd}$$

*Proof.* Consider the family of maps $*\kappa : \mathcal{L}^\phi\Omega^k(G,\mathcal{U}) \to \mathcal{I}^\phi\Omega^k(G,\mathcal{U})$ given by the convolution with a smooth kernel $\kappa$.

Claim 1: For a fixed $k = 0, \ldots, \dim(G)$ the map $*\kappa : \mathcal{L}^\phi\Omega^k(G,\mathcal{U}) \to \mathcal{L}^\phi\Omega^k(G,\mathcal{U})$ is well-defined and continuous.

Let $\gamma > 0$ and $U \in \mathcal{U}$, using Lemma 4.2.1 we have

$$\begin{aligned}
\int_U \phi\left(\frac{|\omega * \kappa|_x}{\gamma}\right) dx &\leq \int_U \phi\left(\int_G \frac{C|\omega|_{xz}}{\gamma}\kappa(z)dz\right) dx \\
&\leq \int_U \phi\left(\int_{x\cdot \mathrm{supp}(\kappa)} \frac{C|\omega|_y}{\gamma} dy\right) dx \\
&\leq \int_U \phi\left(\sum_{U' \in \mathcal{N}_U} \left\|\frac{C\omega|_{U'}}{\gamma}\right\|_{L^1}\right) dx,
\end{aligned}$$

where $\mathcal{N}_U = \{U' \in \mathcal{U} : U' \cap (x \cdot \mathrm{supp}(\kappa)) \neq \emptyset$ for some $x \in U\}$. The bounded geometry implies that there exists $N$ a uniform bound of $\#\mathcal{N}_U$. Using this bound and Jensen's inequality we have

$$\int_U \phi\left(\frac{|\omega * \kappa|_x}{\gamma}\right) dx \leq \frac{\mathrm{Vol}(U)}{\#\mathcal{N}_U} \sum_{U' \in \mathcal{N}_U} \phi\left(\left\|\frac{NC\omega|_{U'}}{\gamma}\right\|_{L^1}\right).$$

Let $V$ a uniform bound for $\mathrm{Vol}(U)$. Because of Lemma 2.5.2, there exists a constant $D$ such that if $\beta$ is in $L^\phi\Omega^k(U)$ with $U \in \mathcal{U}$, then $\|\beta\|_{L^1} \leq D\|\beta\|_{L^\phi}$. Therefore

$$\int_U \phi\left(\frac{|\omega * \kappa|_x}{\gamma}\right) dx \leq \frac{V}{\#\mathcal{N}_U} \sum_{U' \in \mathcal{N}_U} \phi\left(\left\|\frac{DNC\omega|_{U'}}{\gamma}\right\|_{L^\phi}\right).$$

If $\gamma \geq DNC\|\omega|_{U'}\|_{L^\phi}$ for all $U' \in \mathcal{N}_U$ then

$$\int_U \phi\left(\frac{|\omega * \kappa|_x}{\gamma}\right) dx \leq V.$$

By Remark 2.5.1 there exists a constant $\mathcal{C}(V)$ such that

$$\|\omega * \kappa|_U\|_{L^\phi} \leq \mathcal{C}(V)\|\omega * \kappa_U\|_{L^\phi_V} \leq \mathcal{C}(V)DNC \sum_{U' \in \mathcal{N}_U} \|\omega|_{U'}\|_{L^\phi}.$$

Denote $L = \mathcal{C}(V)DNC$ and take $\gamma > 0$,

$$\begin{aligned}\sum_{U\in\mathcal{U}} \phi\left(\frac{\|\omega * \kappa|_U\|_{L^\phi}}{\gamma}\right) &\leq \sum_{U\in\mathcal{U}} \phi\left(\frac{L}{\gamma}\sum_{U'\in\mathcal{N}_U} \|\omega|_{U'}\|_{L^\phi}\right)\\ &\leq \sum_{U\in\mathcal{U}} \frac{1}{\#\mathcal{N}_U} \sum_{U'\in\mathcal{N}_U} \phi\left(\frac{NL}{\gamma}\|\omega|_{U'}\|_{L^\phi}\right).\end{aligned}$$

Let $R > 0$ be such that for all $U' \in \mathcal{U}$, $U' \in \mathcal{N}_U$ for at most $R$ open sets $U \in \mathcal{U}$. Then

$$\sum_{U\in\mathcal{U}} \phi\left(\frac{\|\omega * \kappa|_U\|_{L^\phi}}{\gamma}\right) \leq \sum_{U\in\mathcal{U}} R\phi\left(\frac{NL}{\gamma}\|\omega|_{U'}\|_{L^\phi}\right).$$

This means that, if $\theta(U) = \|\omega|_U\|_{L^\phi}$ and $\vartheta(U) = \|\omega * \kappa|_U\|_{L^\phi}$, then

$$\|\vartheta\|_{\ell^\phi} \leq NL\|\theta\|_{\ell^{R\phi}} \preceq \|\theta\|_{\ell^\phi}.$$

Using the same argument with $d(\omega*\kappa) = d\omega*\kappa$ we can conclude that $|\omega*\kappa|_{\mathcal{L}^\phi} \preceq |\omega|_{\mathcal{L}^\phi}$, which finishes the proof of Claim 1.

Claim 2: Let $k = 0, \ldots, \dim(G)$. The map $*\kappa : \mathcal{L}^\phi\Omega^k(G,\mathcal{U}) \to L^\phi\Omega^k(G)$ is well-defined and continuous.

As above, if $\gamma > 0$ we have

$$\begin{aligned}\int_G \phi\left(\frac{|\omega * \kappa|_x}{\gamma}\right) dx &\leq \int_G \phi\left(\sum_{U'\in\mathcal{N}_U} \left\|\frac{C\omega|_{U'}}{\gamma}\right\|_{L^1}\right) dx\\ &\leq \sum_{U\in\mathcal{U}} \frac{\mathrm{Vol}(U)}{\#\mathcal{N}_U} \sum_{U'\in\mathcal{N}_U} \phi\left(\left\|\frac{DNC\omega|_{U'}}{\gamma}\right\|_{L^phi}\right)\\ &\leq \sum_{U\in\mathcal{U}} VR\phi\left(\left\|\frac{L\omega|_U}{\gamma}\right\|_{L^\phi}\right).\end{aligned}$$

Using again the notation $\theta(U) = \|\omega|_U\|_{L^\phi}$, we have $\|\omega\|_{L^\phi} \leq L\|\theta\|_{\ell^{VR\phi}} \preceq \|\theta\|_{\ell^\phi}$. Doing the same with the derivative we conclude the Claim 2.

Claims 1 and 2 imply that $*\kappa : \mathcal{L}^\phi\Omega^k(G,\mathcal{U}) \to \mathcal{I}^\phi\Omega^k(G)$ is well-defined and continuous. Furthermore, by Proposition 4.2.2 we know that $*\kappa$ commutes with the derivative.

We will define a family of continuous maps $h : \mathcal{L}^\phi\Omega^k(G,\mathcal{U}) \to \mathcal{L}^\phi\Omega^{k-1}(G,\mathcal{U})$ such that

$$\begin{cases} h(df) = f - i(f*\kappa) & \text{if } f \in \mathcal{L}^\phi\Omega^0(G,\mathcal{U}) \\ h(d\omega) + dh(\omega) = \omega - i(\omega*\kappa) & \text{if } \omega \in \mathcal{L}^\phi\Omega^k(G,\mathcal{U}),\ k \geq 1. \end{cases} \tag{4.2}$$

If $h$ maps continuously $\mathcal{I}^\phi\Omega^k(G,\mathcal{U})$ on $\mathcal{I}^\phi\Omega^{k-1}(G,\mathcal{U})$ for every $k \geq 1$ we also have

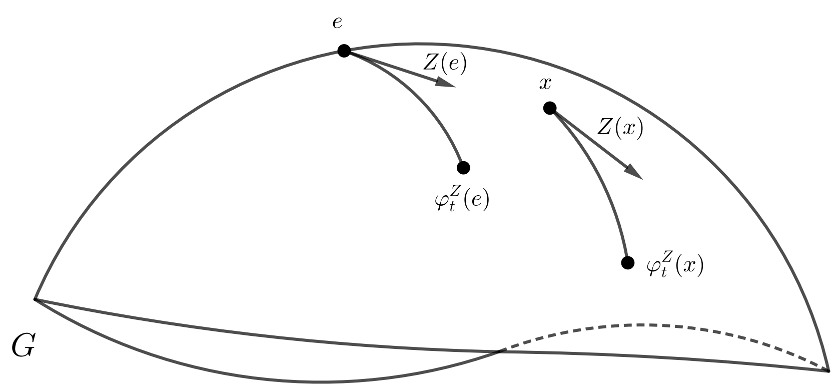
$$\begin{cases} h(df) = f - (if) * \kappa & \text{if } f \in \mathcal{I}^\phi\Omega^0(G,\mathcal{U}) \\ h(d\omega) + dh(\omega) = \omega - (i\omega) * \kappa & \text{if } \omega \in \mathcal{I}^\phi\Omega^k(G,\mathcal{U}),\ k \geq 1. \end{cases}$$

This implies that $(\mathcal{L}^\phi\Omega^*(G,\mathcal{U}), d)$ and $(\mathcal{I}^\phi\Omega^*(G,\mathcal{U}), d)$ are homotopically equivalent. Notice that the inclusion $i : \mathcal{I}^\phi\Omega^k(G,\mathcal{U}) \to \mathcal{L}^\phi\Omega^k(G,\mathcal{U})$ is clearly continuous.

$$\begin{tikzcd}
\mathcal{L}^\phi\Omega^0(G,\mathcal{U}) \arrow[r, "d"'] \arrow[d, bend right, "*\kappa"'] & \mathcal{L}^\phi\Omega^1(G,\mathcal{U}) \arrow[r, "d"'] \arrow[l, bend right, "h"'] \arrow[d, bend right, "*\kappa"'] & \mathcal{L}^\phi\Omega^2(G,\mathcal{U}) \arrow[r] \arrow[l, bend right, "h"'] \arrow[d, bend right, "*\kappa"'] & \cdots \\
\mathcal{I}^\phi\Omega^0(G,\mathcal{U}) \arrow[r, "d"] \arrow[u, bend right, "i"'] & \mathcal{I}^\phi\Omega^1(G,\mathcal{U}) \arrow[r, "d"] \arrow[l, bend left, "h"] \arrow[u, bend right, "i"'] & \mathcal{I}^\phi\Omega^2(G,\mathcal{U}) \arrow[r] \arrow[l, bend left, "h"] \arrow[u, bend right, "i"'] & \cdots
\end{tikzcd}$$

For every $z \in \operatorname{supp}(\kappa)$ we consider $Z \in \operatorname{Lie}(G)$ a left-invariant vector field such that $\exp(Z) = z$. Let $\varphi_t^Z$ be the flow associated to $Z$. Observe that

$$\varphi_t^Z(x) = L_x\varphi_t^Z(e) = x \cdot \exp(tZ).$$




Given $\omega \in \mathcal{L}^\phi\Omega^k(G,\mathcal{U})$ with $k = 1, \ldots, \dim(G)$, we define

$$h(\omega)_x = -\int_G \left( \int_0^1 (\varphi_t^Z)^* \iota_Z \omega_x dt \right) \kappa(z) dz.$$

This is a differential $k$-form because of Lemma 2.3.2, which also gives us an expression for its derivative.

Claim 3: $h$ and $*\kappa$ verify 4.2.

Take $\omega$ a $k$-form in $\mathcal{L}^\phi\Omega^k(G,\mathcal{U})$, then

$$h(d\omega)+dh(\omega) = -\int_G \left(\int_0^1 (\varphi_t^Z)^*\iota_Z d\omega \; dt\right)\kappa(z)dz - d\int_G \left(\int_0^1 (\varphi_t^Z)^*\iota_Z\omega \; dt\right)\kappa(z)dz$$
$$= -\int_G \left(\int_0^1 (\varphi_t^Z)^*(\iota_Z d + d\iota_Z)\omega \; dt\right)\kappa(z)dz.$$

Recall the Cartan formula $L_Z\omega = \iota_Z d\omega + d\iota_Z\omega$ and the identity $\frac{\partial}{\partial t}(\varphi_t^Z)^*\omega = (\varphi_t^Z)^*L_Z\omega$. Then

$$\begin{aligned} h(d\omega)+dh(\omega) &= -\int_G \left(\int_0^1 (\varphi_t^Z)^* L_Z\omega \; dt\right)\kappa(z)dz \\ &= -\int_G \left(\int_0^1 \frac{\partial}{\partial t}(\varphi_t^Z)^*\omega \; dt\right)\kappa(z)dz \\ &= -\int_G ((\varphi_t^Z)^*\omega - \omega)\kappa(z)dz = \omega - \omega * \kappa. \end{aligned}$$

Now consider $f \in \mathcal{L}^\phi\Omega^0(G,\mathcal{U})$, then

$$h(df)_x = -\int_G \left(\int_0^1 df_{x\cdot\exp(tZ)}(Z(x))dt\right)\kappa(z)dz.$$

Let $\alpha : [0,1] \to G$ the curve $\alpha(t) = x\cdot\exp(tZ)$. We have $\alpha'(t) = Z(\alpha(t))$, then

$$(f\circ\alpha)'(t) = d_{\alpha(t)}f(\alpha'(t)) = df_{x\cdot\exp(tZ)}(Z(\alpha(t))).$$

Therefore

$$\begin{aligned} h(df)_x &= -\int_G \left(\int_0^1 (f\circ\alpha)'(t)dt\right)\kappa(z)dz \\ &= \int_G (f(\alpha(1)) - f(\alpha(0)))\kappa(z)dz \\ &= \int_G (f(xz) - f(x))\kappa(z)dz \\ &= f(x) - f*\kappa(x). \end{aligned}$$

Claim 4: $h : \mathcal{L}^\phi\Omega^k(G,\mathcal{U}) \to \mathcal{L}^\phi\Omega^{k-1}(G,\mathcal{U})$ is well-defined and continuous for every $k = 1,\ldots,\dim(G)$.

First we estimate the operator norm of $h\omega$ at a point $x \in G$. Consider $v_1,\ldots,v_{k-1} \in T_xG$, then

$$|h(\omega)_x(v_1,\ldots,v_{k-1})| = \left|\int_G \left(\int_0^1 \omega_{\varphi_t^Z(x)}(Z(\varphi_t^Z(x)), d_x\varphi_t^Z(v_1),\ldots,d_x\varphi_t^Z(v_{k-1}))dt\right)\kappa(z)dz\right|$$
$$\leq \int_G \left(\int_0^1 |\omega_{\varphi_t^Z(x)}(Z(\varphi_t^Z(x)), d_xR_{\exp(tZ)}(v_1),\ldots,d_xR_{\exp(tZ)}(v_{k-1}))|dt\right)\kappa(z)dz$$

As in the proof of Lemma 4.2.1 we have a uniform bound $|d_x R_{\exp(tZ)}| \leq M$ for all $z \in \operatorname{supp}(\kappa)$. Moreover, since left-invariant fields have constant norm we can write $\|Z(y)\|_y = C$ for every $y \in G$. Then if $\|v_1\|_x = \ldots = \|v_{k-1}\|_x = 1$,

$$|h(\omega)_x(v_1, \ldots, v_{k-1})| \leq \int_G \left( \int_0^1 CM^{k-1} |\omega|_{\varphi_t^Z(x)} dt \right) \kappa(z) dz,$$

which implies

$$|h(\omega)|_x \leq \int_G \left( \int_0^1 CM^{k-1} |\omega|_{\varphi_t^Z(x)} dt \right) \kappa(z) dz. \tag{4.3}$$

Using (4.3) and Jensen's inequality we obtain

$$\phi\left( \frac{|h(\omega)|_x}{\gamma} \right) \leq \int_G \left( \int_0^1 \phi\left( \frac{CM^{k-1}}{\gamma} |\omega|_{\varphi_t^Z(x)} \right) dt \right) \kappa(z) dz. \tag{4.4}$$

For $U \in \mathcal{U}$ denote $\theta(U) = \|\omega|_U\|_{L^\phi}$ and $\vartheta(U) = \|h\omega|_U\|_{L^\phi}$. If $\gamma > 0$ we have

$$\begin{aligned}\int_U \phi\left( \frac{|h(\omega)|_x}{\gamma} \right) dx &\leq \int_U \int_G \left( \int_0^1 \phi\left( \frac{CM^{k-1}}{\gamma} |\omega|_{\varphi_t^Z(x)} dt \right) \right) \kappa(z) dz dx \\ &= \int_G \left( \int_0^1 \int_U \phi\left( \frac{CM^{k-1}}{\gamma} |\omega|_{\varphi_t^Z(x)} \right) dx dt \right) \kappa(z) dz.\end{aligned}$$

The identity $d_x \varphi_t^Z = d_x R_{\exp(tZ)}$ allows us to find $m > 0$ such that $m < |Jac_x(\varphi_t^Z)|$ for all $z \in \operatorname{supp}(\kappa)$. Then

$$\begin{aligned}\int_U \phi\left( \frac{|h(\omega)|_x}{\gamma} \right) dx &\leq \int_G \left( \int_0^1 \int_{E(U)} \frac{1}{m} \phi\left( \frac{CM^{k-1}}{\gamma} |\omega|_y \right) dy dt \right) \kappa(z) dz \\ &= \frac{1}{m} \int_{E(U)} \phi\left( \frac{CM^{k-1}}{\gamma} |\omega|_y \right) dy,\end{aligned}$$

where $E(U)$ is a neighborhood of $U$ with uniform radius (independent of $U$) such that $\varphi_t^Z(x) \in E(U)$ for all $z \in \operatorname{supp}(\kappa)$ and $x \in U$. Consider

$$\mathcal{V}_U = \{V \in \mathcal{U} : V \cap E(U) \neq \emptyset\},$$

then if $\gamma \geq CM^{k-1} \max\{\|\omega|_V\|_{L^{\frac{S}{m}\phi}} : V \in \mathcal{V}_U\}$, where $S \geq \#\mathcal{V}_U$ for all $U \in \mathcal{U}$,

$$\int_U \phi\left( \frac{|h(\omega)|_x}{\gamma} \right) dx \leq 1,$$

which implies

$$\|h(\omega)|_U\|_{L^\phi} \leq CM^{k-1} \max\{\|\omega|_V\|_{L^{\frac{S}{m}\phi}} : V \in \mathcal{V}_U\} \leq \mathcal{M} \sum_{V \in \mathcal{V}_U} \|\omega|_V\|_{L^\phi}$$

for some constant $\mathcal{M}$ that does not depend on $U$. Therefore

$$\begin{aligned}\sum_{U\in\mathcal{U}}\phi\left(\frac{\|h(\omega)|_U\|_{L^\phi}}{\gamma}\right) &\leq \sum_{U\in\mathcal{U}}\phi\left(\mathcal{M}\sum_{V\in\mathcal{V}_U}\frac{\|\omega|_V\|_{L^\phi}}{\gamma}\right)\\ &\leq \sum_{U\in\mathcal{U}}\sum_{V\in\mathcal{V}_U}\frac{1}{\#\mathcal{V}_U}\phi\left(\frac{S\mathcal{M}\|\omega|_V\|_{L^\phi}}{\gamma}\right)\\ &\leq \sum_{U\in\mathcal{U}}\mathcal{N}\phi\left(\frac{S\mathcal{M}\|\omega|_V\|_{L^\phi}}{\gamma}\right),\end{aligned}$$

where $\mathcal{N}\geq\#\{U\in\mathcal{U}:V\in\mathcal{V}_U\}$ for all $V\in\mathcal{U}$. From here we obtain

$$\|\vartheta\|_{\ell^\phi}\leq S\mathcal{M}\|\theta\|_{\ell^{\mathcal{N}\phi}}\preceq\|\theta\|_{\ell^\phi}.$$

Using the identity $dh(\omega)=\omega-i(\omega*\kappa)$ and the above estimate we obtain

$$|h(\omega)|_{\mathcal{L}^\phi}\preceq|\omega|_{\mathcal{L}^\phi}.$$

Claim 5: The map $h:L^\phi\Omega^k(G)\to L^\phi\Omega^{k-1}(G)$ is well-defined and continuous for every $k=1,\ldots,\dim(G)$.

Using 4.4 we have

$$\begin{aligned}\int_G\phi\left(\frac{|h(\omega)|_x}{\gamma}\right)dx &\leq \int_G\int_G\left(\int_0^1\phi\left(\frac{CM^{k-1}|\omega|_{\varphi_t^z(x)}}{\alpha}\right)dt\right)\kappa(z)dzdx\\ &\leq \int_G\left(\int_0^1\int_G\frac{1}{m}\phi\left(\frac{CM^{k-1}|\omega|_y}{\gamma}\right)dydt\right)\kappa(z)dz\\ &= \int_G\frac{1}{m}\phi\left(\frac{CM^{k-1}|\omega|_y}{\gamma}\right)dy.\end{aligned}$$

From this we obtain $\|h(\omega)\|_{L^\phi}\preceq\|\omega\|_{L^\phi}$; and using again the equality (4.2) we have $|h(\omega)|_{L^\phi}\preceq|\omega|_{L^\phi}$.

By Claims 4 and 5 we conclude that $h$ is well-defined and continuous from $\mathcal{I}^\phi\Omega^k(G,\mathcal{U})$ to $\mathcal{I}^\phi\Omega^{k-1}(G,\mathcal{U})$.

The same argument works in the relative case, the only thing we have to verify is that the maps $*\kappa$ and $h$ preserve the relative subcomplexes. This is easy using the compactness of $\mathrm{supp}(\kappa)$. $\square$

The proof of Theorem 1.2.9 finishes with the following proposition.

**Proposition 4.2.10.** *The cochain complexes $(\mathcal{I}^\phi\Omega^*(G,\mathcal{U}),d)$, $(L^\phi\Omega^*(G),d)$, and $(L^\phi C^*(G),d)$ are homotopically equivalent. The same result is true for the corresponding relative cochain complexes.*

*Proof.* In this case we consider $*\kappa$ and $h$ defined as in Proposition 4.2.9. We have to prove that they are well-defined and continuous. Identities as (4.2) are clearly satisfied.

$$\begin{tikzcd}
L^\phi C^0(G) \arrow[r, "d"] \arrow[d, bend right, "*\kappa"'] & L^\phi C^1(G) \arrow[r, "d"] \arrow[l, bend right, "h"'] \arrow[d, bend right, "*\kappa"'] & L^\phi C^2(G) \arrow[r, "d"] \arrow[l, bend right, "h"'] \arrow[d, bend right, "*\kappa"'] & \cdots \\
\mathcal{I}^\phi\Omega^0(G,\mathcal{U}) \arrow[r, "d"] \arrow[u, bend right, "i"'] \arrow[d, bend right, "i"'] & \mathcal{I}^\phi\Omega^1(G,\mathcal{U}) \arrow[r, "d"] \arrow[l, bend right, "h"'] \arrow[u, bend right, "i"'] \arrow[d, bend right, "i"'] & \mathcal{I}^\phi\Omega^2(G,\mathcal{U}) \arrow[r, "d"] \arrow[l, bend right, "h"'] \arrow[u, bend right, "i"'] \arrow[d, bend right, "i"'] & \cdots \\
L^\phi\Omega^0(G) \arrow[r, "d"] \arrow[u, bend right, "*\kappa"'] & L^\phi\Omega^1(G) \arrow[r, "d"] \arrow[l, bend right, "h"'] \arrow[u, bend right, "*\kappa"'] & L^\phi\Omega^2(G) \arrow[r, "d"] \arrow[l, bend right, "h"'] \arrow[u, bend right, "*\kappa"'] & \cdots
\end{tikzcd}$$

The map $h : L^\phi C^k(G) \to L^\phi C^{k-1}(G)$ is the continuous extension of $h : L^\phi C^k(G) \to L^\phi C^{k-1}(G)$, then we have that all maps $h$ in the diagram are continuous.

Claim: The map $*\kappa : L^\phi C^k(G) \to \mathcal{I}^\phi\Omega^k(G,\mathcal{U})$ is well-defined and continuous for every $k = 0, \ldots, \dim(G)$. Then so is $*\kappa : L^\phi\Omega^k(G) \to \mathcal{I}^\phi\Omega^k(G,\mathcal{U})$

First of all observe that if $\omega \in L^\phi C^k(G)$ then $\omega * \kappa \in \Omega^k(G)$ by Lemma 4.2.1. Remember the estimate given by Lemma 4.2.1:

$$|\omega * \kappa|_x \leq C|\omega| * \kappa(x).$$

For $\gamma > 0$ we have

$$\begin{aligned}
\int_G \phi\left(\frac{|\omega * \kappa|_x}{\gamma}\right) dx &\leq \int_G \phi\left(\frac{C|\omega| * \kappa(x)}{\gamma}\right) dx \\
&= \int_G \phi\left(\int_G \frac{C|\omega|_{xz}}{\gamma}\kappa(z)dz\right) dx \\
&\leq \int_G\int_G \phi\left(\frac{C|\omega|_{xz}}{\gamma}\right)\kappa(z)dzdx.
\end{aligned}$$

In the last line we use Jensen's inequality. As before we take $m > 0$ with $m < |Jac_x(R_z)|$ for all $x \in G$ and $z \in \mathrm{supp}(\kappa)$, then

$$\begin{aligned}
\int_G \phi\left(\frac{|\omega * \kappa|_x}{\gamma}\right) dx &= \int_G\left(\int_G \phi\left(\frac{C|\omega|_{xz}}{\gamma}\right)\frac{|Jac_x(R_z)|}{|Jac_x(R_z)|}dx\right)\kappa(z)dz \\
&\leq \int_G\left(\int_G \frac{1}{m}\phi\left(\frac{C|\omega|_y}{\gamma}\right)dy\right)\kappa(z)dz \\
&= \int_G \frac{1}{m}\phi\left(\frac{C|\omega|_y}{\gamma}\right)dy.
\end{aligned}$$

If $\gamma \geq C\|\omega\|_{L^{\frac{\phi}{m}}}$ we have

$$\int_G \phi\left(\frac{|\omega * \kappa|_x}{\gamma}\right) dx \leq 1,$$

which implies $\|\omega*\kappa\|_{L^\phi}\leq C\|\omega\|_{L^{\frac{\phi}{m}}}\preceq\|\omega\|_{L^\phi}$. In the same way we have $\|d(\omega*\kappa)\|_{L^\phi}=\|d\omega*\kappa\|_{L^\phi}\preceq\|d\omega\|_{L^\phi}$ and as a conclusion $|\omega*\kappa|_{L^\phi}\preceq|\omega|_{L^\phi}$.

On the other hand we denote $\vartheta(U)=\|\omega*\kappa|_U\|_{L^\phi}$ and estimate

$$\begin{aligned}\int_U\phi\left(\frac{|\omega*\kappa|_x}{\gamma}\right)dx&\leq\int_U\phi\left(\int_G\frac{C|\omega|_{xz}}{\gamma}\kappa(z)dz\right)dx\\&\leq D\phi\left(\int_{E(U)}\frac{C|\omega|_y}{\gamma}dy\right),\end{aligned}$$

where $E(U)$ is a neighborhood of $U$ with radius independent of $U$, and $D$ is a constant (also independent of $U$). We can deduce from here that

$$\|\omega*\kappa|_U\|_{L^\phi}\leq\frac{C}{\phi^{-1}(1/D)}\int_{E(U)}|\omega|_ydy.$$

In order to simplify the notation we write $\mathcal{C}=\frac{C}{\phi^{-1}(1/D)}$. Then

$$\begin{aligned}\sum_{U\in\mathcal{U}}\phi\left(\frac{\|\omega*\kappa|_U\|_{L^\phi}}{\gamma}\right)&\leq\sum_{U\in\mathcal{U}}\phi\left(\frac{\mathcal{C}}{\gamma}\int_{E(U)}|\omega|_ydy\right)\\&\leq\sum_{U\in\mathcal{U}}\frac{1}{\operatorname{Vol}(E(U))}\int_{E(U)}\phi\left(\frac{\mathcal{C}\operatorname{Vol}(E(U))|\omega|_y}{\gamma}\right)dy.\end{aligned}$$

Using that $\{E(U):U\in\mathcal{U}\}$ is a uniformly locally finite covering such that $\operatorname{Vol}(E(U))$ is bounded from above and below far from zero, we can find a uniform constant $L$ such that

$$\sum_{U\in\mathcal{U}}\frac{1}{\operatorname{Vol}(E(U))}\int_{E(U)}\phi\left(\frac{\mathcal{C}\operatorname{Vol}(E(U))|\omega|_y}{\gamma}\right)dy\leq\int_G L\phi\left(\frac{L|\omega|_y}{\gamma}\right)dy.$$

This proves that $\|\vartheta\|_{\ell^\phi}\preceq\|\omega\|_{L^\phi}$. Doing the same for the derivative we obtain $|\omega*\kappa|_{\mathcal{L}^\phi}\preceq|\omega|_{L^\phi}$, that finish de proof of the Claim.

As in Proposition 4.2.9 the relative case follows from the previous argument. $\square$

Observe that the previous proposition has a consequence that is not trivial: to study the (relative) $L^\phi$-cohomology in the case of Lie groups it is enough to consider differential forms. This result is proved in a more general case in [KP15] for the non-relative version.

# Chapter 5

# Some questions

## 5.1 Dependence on the boundary point

Example 3.1.4 shows a uniformly contractible and Gromov-hyperbolic simplicial complex $X$ with bounded geometry for which $\ell^p H^1(X, \xi_0)$ is not isomorphic to $\ell^p H^1(X, \xi)$, where $\xi_0$ and $\xi$ are two different points in $\partial X$.

This kind of result allows to conclude that there is no quasi-isometry from the space to itself whose boundary map sends $\xi_0$ on $\xi$. In the case studied the topology of the boundary gives a more direct proof of this fact, but it is not always so easy.

In the case of a Heintze group $G = N \rtimes_\alpha \mathbb{R}$, whose boundary is $\partial G = N \cup \{\infty\}$, we have that $L^p H^k(G, \xi)$ is isomorphic to $L^p H^k(G, \eta)$ for every $k \in \mathbb{N}$ if $\xi, \eta \in N \subset \partial G$ because $\mathrm{QI}(G)$ acts transitively on $N$ by boundary maps. It makes sense to ask the following question:

**Question 5.1.1.** *Is $L^p H^k(G, \xi)$ isomorphic to $L^p H^k(G, \infty)$ if $\xi \neq \infty$?*

A negative answer to this question for some $k$ and $p$ would imply that $\infty$ is fixed by the group $QI(G)$. Observe that the question can be formulated also for relative Orlicz cohomology.

It is known that the boundary point $\infty$ is fixed by $QI(G)$ if $G$ is not of Carnot type (see [Car16]) and it is easy to see that $QI(G)$ acts transitively on $\partial G$ if $G$ is a symmetric space. The non-symmetric Carnot type case reminds open.

## 5.2 Relative Orlicz cohomology of Heintze groups

Consider the family of doubling Young functions

$$\phi_{p,\kappa}(t) = \frac{|t|^p}{\log(e + |t|^{-1})^\kappa},$$

with $p \geq 1$ and $\kappa \in \mathbb{R}$. We can call $p$ the *main exponent* and $\kappa$ the *logarithmic exponent* of the Young function.

**Example 5.2.1.** Consider $G = \mathbb{R}^{n-1} \rtimes_\alpha \mathbb{R}$ a purely real Heintze group with $\alpha$ diagonalizable and denote by $\lambda_1 \leq \cdots \leq \lambda_{n-1}$ the eigenvalues of $\alpha$. The numbers $w_k$ are defined as in Section 3.4.

Using similar methods as we use in Section 3.4 we can prove that:

(i) $L^{\phi_{p,\kappa}} H^k(G, \infty) = 0$ for $p > \frac{\operatorname{tr}(\alpha)}{w_{k-1}}$ and all $\kappa$; and

(ii) $L^{\phi_{p,\kappa}} H^k(G, \infty) \neq 0$ for $p \in \left( \frac{\operatorname{tr}(\alpha)}{w_k}, \frac{\operatorname{tr}(\alpha)}{w_{k-1}} \right]$ and all $\kappa$.

As one can observe in the previous example, the logarithmic exponent of the function $\phi_{p,\kappa}$ seems to be negligible. Looking at the computations it seems to be related to the fact that $\alpha$ has not Jordan blocks (with size bigger than one).

We would like to get numerical quasi-isometry invariants related to the sizes of the Jordan blocks of the derivation $\alpha$ defining a purely real Heintze group $G = \mathbb{R}^{n-1} \rtimes_\alpha \mathbb{R}$ (or more in general $G = N \rtimes_\alpha \mathbb{R}$). We think that critical logarithmic exponents could give us such invariants. Here the notion of *critical logarithmic exponent* is a bit vague, we can think of it as an exponent $\kappa_c$ such that there exist fixed $p > 1$ and $k \in \mathbb{N}$, and a property satisfied by $L^{\phi_{p,\kappa}} H^k(G, \infty)$ for $\kappa$ in an interval of the form $(\kappa_0, \kappa_c)$, but not satisfied by the cohomology space for $\kappa$ in an interval $(\kappa_c, \kappa_1)$. In the case of Example 5.2.1 we can say that $\frac{\operatorname{tr}(\alpha)}{w_{k-1}}$ is a *critical main exponent* for the property of being zero as vector space, but there is not critical logarithmic exponent for the same property.

Being more ambitious, we could try to answer Question 1.2.11 using these methods. To justify that this question makes sense and that it is related to Conjecture 1.2.6 we can look at the following proposition:

**Proposition 5.2.2.** *Consider two purely real Heintze groups $G_1 = N_1 \rtimes_{\alpha_1} \mathbb{R}$ and $G_2 = N_2 \rtimes_{\alpha_2} \mathbb{R}$. If $G_1$ and $G_2$ are isomorphic, then there exists $\lambda > 0$ such that $\alpha_1$ and $\lambda\beta_2$ have the same Jordan form.*

The Lie algebra of a Heintze group $N \rtimes_\alpha \mathbb{R}$ is the direct sum of $\mathfrak{n}$ and $\mathbb{R}$ (where $\mathfrak{n}$ is the Lie algebra of $N$) with the Lie bracket defined by

$$[X + T, Y + S] = [X, Y] + T\alpha(Y) - S\alpha(X). \tag{5.1}$$

In the right side of the equality (5.1) $[X, Y]$ indicates the Lie bracket on $\mathfrak{n}$. We denote this Lie algebra by $\mathfrak{n} \rtimes_\alpha \mathbb{R}$.

**Lemma 5.2.3.** *In the same hypotheses of Proposition 5.2.2 denote by $\mathfrak{n}_i$ the Lie algebra of $N_i$ for $i = 1, 2$. Then $G_1$ and $G_2$ are isomorphic if, and only if, there exist an isomorphism $\gamma : \mathfrak{n}_1 \to \mathfrak{n}_2$, $X \in \mathfrak{n}_2$ and $\lambda > 0$ such that*

$$\gamma \circ \alpha_1 \circ \gamma^{-1} - \lambda\alpha_2 = ad_X \tag{5.2}$$

*Proof.* ($\Rightarrow$) Since $G_1$ and $G_2$ are isomorphic, there exist an isomorphism

$$\Phi : \mathfrak{n}_1 \rtimes_{\alpha_1} \mathbb{R} \to \mathfrak{n}_2 \rtimes_{\alpha_2} \mathbb{R}.$$

We denote $\gamma = \Phi|_{\mathfrak{n}_1}$ and $\Phi(1) = X + \lambda$, with $X \in \mathfrak{n}_2$ and $\lambda > 0$. For all $Y \in \mathfrak{n}_1$ we have

$$\Phi([1, Y]) = \Phi(\alpha_1(Y)) = \gamma \circ \alpha_1(Y)$$

and

$$\Phi([1, Y]) = [\Phi(1), \Phi(Y)] = [X + \lambda, \gamma(Y)] = ad_X \circ \gamma(Y) + \lambda\alpha_2 \circ \gamma(Y).$$

From here follows (5.2).

($\Leftarrow$) To prove this part observe that the linear map $\Phi : \mathfrak{n}_1 \rtimes_{\alpha_1} \mathbb{R} \to \mathfrak{n}_2 \rtimes_{\alpha_2} \mathbb{R}$ verifying $\gamma = \Phi|_{\mathfrak{n}_1}$ and $\Phi(1) = X + \lambda$, is an isomorphism of Lie algebras. $\square$

*Proof of Proposition 5.2.2.* By (5.2) the derivation $\gamma\alpha_1\gamma^{-1} - \lambda\alpha_2$ is nilpotent and thus $\alpha_1$ and $\lambda\alpha_2$ have the same positive eigenvalues $\lambda_1 < \cdots < \lambda_d$ with the same multiplicity. Without loss of generality we can assume that $\mathfrak{n}_1 = \mathfrak{n}_2$, $\gamma = \mathrm{Id}$ and $\lambda = 1$.

We prove the proposition by induction on $d$, the number of eigenvalues. Observe that if $d = 1$, then $\mathfrak{n}_1$ must be abelian; hence $ad_X = 0$.

Now consider $d \geq 2$. We assume that the proposition is true for $d - 1$. Denote by $D_1$ and $D_2$ the diagonal part of $\alpha_1$ and $\alpha_2$ respectively, they are also derivations (see [CS17, Section 2]).

Consider $V_d, W_d \subset \mathfrak{n}_1$ the generalized eigenspaces associated to $\lambda_d$ for $\alpha_1$ and $\alpha_2$ respectively.

<u>Claim</u>: $V_d = W_d \subset Z(\mathfrak{n}_1)$, where $Z(\mathfrak{n}_1)$ is the center of the Lie algebra $\mathfrak{n}_1$.

Let $X_d \in V_d$ and $X_i$ another eigenvector of $D_1$ associated to $\lambda_i$. Then

$$D_1([X_d, X_i]) = (\lambda_i + \lambda_d)[X_d, X_i].$$

Since $\lambda_d$ is the biggest eigenvalue of $D_1$ we have that $[X_d, X_i] = 0$. This implies that $X_d \in Z(\mathfrak{n}_1)$.

The same can be done for $W_d$ using $D_2$. By (5.2) we have $\alpha_1|_{V_d} = \alpha_2|_{V_d}$ and $\alpha_1|_{W_d} = \alpha_2|_{W_d}$ and as a consequence $V_d = W_d$.

Since $V_d$ is invariant by $\alpha_1$ and $\alpha_2$ and it is contained in $Z(\mathfrak{n}_1)$, we can consider on the Lie algebra $\mathfrak{n}_1/V_d$ the derivations induced by $\alpha_1$, $\alpha_2$ and $ad_X$, denoted by $\overline{\alpha_1}$, $\overline{\alpha_2}$ and $\overline{ad_X}$. Observe that $\overline{ad_X} = ad_{\overline{X}}$, where $\overline{X} = X + V_d$. We also have the identity

$$\overline{\alpha_1} - \overline{\alpha_2} = ad_{\overline{X}}.$$

Since $\overline{\alpha_1}$ and $\overline{\alpha_2}$ have positive eigenvalues $\lambda_1 < \cdots < \lambda_{d-1}$ with the same multiplicity, both derivations have the same Jordan form by the induction hypotheses. Combining this with the fact that $\alpha_1|_{V_d} = \alpha_2|_{V_d}$, we conclude that $\alpha_1$ and $\alpha_2$ have the same Jordan form. $\square$

It is important to have on mind that the converse of Proposition 5.2.2 is not true (see [CS17]).

## 5.3 Equivalence of Orlicz cohomology on a more general case

Our main motivation to study the Orlicz cohomology is to use it to obtain information about the large scale geometry of Heintze groups. For this purpose the generality in which we state Theorem 1.2.3 is enough. However, it could be interesting to have a more general result. Thinking about the $L^p$-case we can ask:

**Question 5.3.1.** *Is Theorem 1.2.3 true for a complete Riemannian manifold with bounded geometry?*

# Bibliography


[Att94] O. Attie; Quasi-isometry classification of some manifolds of bounded geometry. *Mathematische Zeitschrif*, (1994). Doi:10.1007/BF02572337.

[BHK01] M. Bonk, J. Heinonen, and P. Koskela. Uniformizing Gromov-hyperbolic spaces. *Astérisque 270. Société Mathématique de France*, (2001). Zbl 0970.30010 MR 1829896.

[BK12] M. Bourdon and B. Kleiner; Some applications of $l_p$-cohomology to boundaries of Gromov hyperbolic spaces. *Groups, geometry and dynamics* (2012). Doi:10.4171/GGD/318.

[Bou16] M. Bourdon; Cohomologie $\ell_p$ en degrés supérieurs et dimension conforme. *Annales de l'Institut Fourier*, 66 n°3 (2016).

[BP03] M. Bourdon, and H. Pajot; Cohomologie $\ell_p$ et espaces de Besov. *Journal für die Reine und Angewandte Mathematik. [Crelle's Journal]*, (2003). Doi:10.1515/crll.2003.043.

[BR19] M. Bourdon and B. Remy; Quasi-isometric invariance of continuous group $L^p$-cohomology and first applications to vanishings. *arXiv* (2019).

[Car16] M. Carrasco; Orlicz spaces and the large scale geometry of Heintze groups. *Mathematische Annalen* (2016). Doi:10.1007/s00208-016-1430-1.

[Cor18] Y. Cornulier; On the quasi-isometric classification of focal hyperbolic groups. *New Directions in Locally Compact Groups* 275–342 (2018).

[CS17] M. Carrasco and E. Sequeira; On quasi-isometry invariants associated to a Heintze group. *Geometriae Dedicata*, (2017). Doi:10.1007/s10711-016-0214-9.

[CT11] Y. Cornulier and R. Tessera; Contracting automorphisms and $L^p$-cohomology in degree one. *Ark. Mat.*, 49(2) 295–324, 2011.

[Do92] M. Do Carmo; *Riemannian Geometry.* Mathematics: Theory and Applications, Birkhäuser 1992.

[Ele97] G. Elek; The $\ell_p$-cohomology and the conformal dimension of hyperbolic cones. *Geom. Ded.* **8** 263-279 (1997).

[Gen14] L. Genton; *Scaled Alexander-Spanier cohomology and Lqp cohomology for metric spaces.* Thesis 2014.

[GdlH90] E. Ghys and P. De la Harpe ; *Sur les groupes hyperboliques d'après Mikhael Gromov*. Progress in Mathematics 1990.

[GHL90] S. Gallot, D. Hulin and J. Lafontaine; *Riemannian Geometry*. Sringer-Verlag 1990. Doi:10.1007/978-3-642-97242-3.

[GK19] V. Gol'dshtein and Y. Kopylov; Some calculations of Orlicz cohomology and Poincaré-Sobolev-Orlicz inequalities. *Sib. Elektron. Mat. Izv*, **16** (2019). Doi:10.33048/semi.2019.16.075.

[GKS84] V. Gol'dshtein, V. Kuzminov, I. Shvedev; A property of De Rham regularisation operators. *Siberian Math. Journal* 25 No 2 (1984). Doi:10.1007/BF00971462.

[GKS86] V. Gol'dshtein, V. Kuzminov, I. Shvedev; Dual spaces of Spaces of Differential Forms. *Siberian Math. Journal* 54 No 1 (1986).

[GKS87] V. Gol'dshtein, V. Kuzminov, I. Shvedev; $L^p$-cohomology of Riemannian manifolds. *Issledovania po Geometrii i Matematicheskomu Analizu (Trudy Inst. Mat. Novosibirsk)*. Nauka, Novosibirsk, 1987, 7, 101-116.

[GKS88] V. Gol'dshtein, V. Kuzminov, I. Shvedev; The de Rham isomorphismes of Lp-cohomology of non compact Riemannian manifolds (Russian). *Sibirsk Mat. Zh., 1988, 29, 2, 33-44. (Engl. transl.: Siberian Math. J.*, 29, (1988), 2, 190-197.

[Gro93] Gromov, Mikhail; *Asymtotic invariants for infinite groups*. G. A. Niblo and M. A. Roller, eds., London Math. Soc. Lect. Note Ser. 182 (1993).

[GT98] V. Gol'dshtein, M. Troyanov; The $L_{p,q}$-cohomology of SOL. *Annales de la Faculté des sciences de Toulouse* 7 no. 4, p. 687-698 (1998).

[GT06] V. Gol'dshtein, M. Troyanov; Sobolev inequiality for differential forms and $L_{p,q}$-cohomology; *Journal of Geom. Anal.* 16 No 4, 597-631, (2010).

[GT10] V. Gol'dshtein and M. Troyanov; A short proof of the Hölder-Poincaré Duality for L p-Cohomology. *Rendiconti del Seminario Matematico della Università di Padova*. 124, (2010). Doi:10.4171/RSMUP/124-11.

[Hei01] J. Heinonen; *Lectures on analysis on metric spaces*. Universitext, Sprinberg-Verlag New York, (2001). Doi: 10.1007/978-1-4613-0131-8.

[Htz74] E. Heintze; On homogeneous manifolds of negative curvature. *Mathematische Annalen*, (1974). Doi:10.1007/BF01344139.

[Kop17] Y. Kopylov; Orlicz spaces of differential forms on Riemannian manifolds: duality and cohomology. *Problemy Analiza*, 6(2), 57-80. (2017) https://doi.org/10.15393/j3.art.2017.3850

[KP15] Y. Kopylov and R. Panenco; De Rham regularization operators in orlicz spaces of diferential forms on Riemannian manifolds. *Sib. Elektron. Mat. Izv*, **12** (2015). Doi:10.17377/semi.2015.12.030.

[LX16] E. Le Donne and X. Xie; Rigidity of fiber-preserving quasisymmetric maps. *Rev. Mat. Iberoam.* 32 (2016), 1407-1422. Doi: 10.4171/RMI/923

[Pan88] P. Pansu; Cohomologie $L^p$ des variétés à courbure négative, cas du degré 1. *Rend. Sem. Mat. Univ. Pol. Torino,* Spacial Issue (1989). Conference on Partial Differential Equations and Geometry (Torino, 1988).

[Pan89] P. Pansu; Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math. (2)*, (1989); Doi:10.2307/1971484.

[Pan95] P. Pansu; Cohomologie $L^p$:invariance sous quasiisométries. (1995). https://www.math.u-psud.fr/ pansu/liste-prepub.html

[Pan08] P. Pansu; Cohomologie $L^p$ et pincement. *Commentarii Mathematici Helvetici. A Journal of the Swiss Mathematical Society*, (2008); Doi:10.4171/CMH/126.

[RR91] M. Rao and Z. Ren; *Teory of Orlicz spaces.* Volume 146 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, (1991).

[SX12] N. Shanmugalingam and X. Xie; A rigidity property of some negatively curved-solvable Lie groups. *omment. Math. Helv.*, 7(4):805–823, (2012).

[Whi57] H. Whitney; *Geometric integration theory*; Dover Publications, Inc. 1957.

[Xie14] X. Xie; Large scale geometry of negatively curved $\mathbb{R}^n \rtimes \mathbb{R}$. *Geometry & Topology*, **21** 3 (2014); Doi:10.2140/gt.2014.18.831.

[Xie15a] X. Xie; Quasiisometries of negatively curved homogeneous manifolds associated with Heisenberg groups. Journal of Topology, **8** 1, 247–266, (2015). Doi:10.1112/jtopol/jtu023.

[Xie15b] X. Xie; Rigidity of quasi-isometries of HMN associated with non-diagonalizable derivation of the Heisenberg algebra. *Q. J. Math.*, 66(1):353–367, (2015).